MATH 505a **QUALIFYING EXAM** **January 2025**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

(1) Let (X, Y) be the coordinates of a point uniformly selected from the unit square $(0,1)^2$. Express the conditional expectation $\mathbb{E}[X|XY]$ as an elementary function of X and Y. Note that $\mathbb{E}[X|XY]$ is a random variable and not a number.

(2) Suppose that $0 \le X \le N$ are integer-valued random variables with the following joint distribution. For $\lambda > 0$ and $p \in (0,1)$ fixed we have that

$$\mathbb{P}(X = k, N = n) = e^{-\lambda}\frac{\lambda^n}{n!} \cdot \binom{n}{k} p^k (1-p)^{n-k} \text{ for } 0 \le k \le n.$$

(a) Express $\mathbb{P}(X = k)$ as an elementary function in λ, p and $k \ge 0$.
(b) Express $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ as elementary functions in λ and p .
(c) Express $\mathbb{E}[z^N|X = k]$ as an elementary function in z, λ, p and $k \ge 0$.
(d) Find a constant c_p, depending on p alone, such that $\lambda^{-1/2} \cdot (X - c_p\lambda)$ converges in distribution as $\lambda \to \infty$, and find the limit. You may assume that $\lambda \in \mathbb{N}$.

(3) Let $n \ge 5$, and $A_1, \dots, A_n$ be i.i.d. random capital letters. Suppose that $p = \mathbb{P}(A_i = \text{D})$ and $q = \mathbb{P}(A_i = \text{A})$. Let S_n be the number of times "DAD" appears as 3 consecutive letters in the word "$A_1 A_2 \cdots A_n$". For example, if $n = 10$, and the word is "ADDDDDDDDD", then $S_{10} = 0$ (there is no wrap-around), while if the word is "BRAVEDADAD", then $S_{10} = 2$.

(a) Find $\mathbb{E}[S_n]$ as a function of p, q, n.
(b) Find $\mathrm{Var}(S_n)$ as a function of p, q, n.