Qualifying Exam: Applied Probability

Unofficial solutions by Alex Fu

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1. Let X_1 and X_2 be independent random variables with distributions Poisson(λ_1) and Poisson(λ_2) respectively.

   a. Find $\mathbb{P}(X_1 = k \mid X_1 + X_2 = n)$ for $0 \le k \le n$.

   Solution. By direct computation,

$$\begin{aligned}\mathbb{P}(X_1 = k \mid X_1 + X_2 = n) &= \frac{\mathbb{P}(X_1 = k, X_1 + X_2 = n)}{\mathbb{P}(X_1 + X_2 = n)} \\ &= \frac{\mathbb{P}(X_1 = k, X_2 = n-k)}{\sum_{\ell=0}^n \mathbb{P}(X_1 = \ell, X_2 = n-\ell)} \\ &= \frac{\lambda_1^k \lambda_2^{n-k} \cancel{e^{-\lambda_1} e^{-\lambda_2}} / (k!(n-k)!)}{\sum_{\ell=0}^n \lambda_1^\ell \lambda_2^{n-\ell} \cancel{e^{-\lambda_1} e^{-\lambda_2}} / (\ell!(n-\ell)!)} \\ &= \frac{\binom{n}{k} \lambda_1^k \lambda_2^{n-k}}{\sum_{\ell=0}^n \binom{n}{\ell} \lambda_1^\ell \lambda_2^{n-\ell}} \\ &= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k}.\end{aligned}$$

   Remark. In other words, the conditional distribution of X_1 given $\{X_1 + X_2 = n\}$ is Binomial(n, p), where $p = \lambda_1/(\lambda_1 + \lambda_2)$. By symmetry, the conditional distribution of X_2 given $X_1 + X_2 = n$ is Binomial($n, 1-p$).

   b. Find $\mathbb{E}(X_1^2 + X_2^2 \mid X_1 + X_2 = n)$.

   Solution. By part (a), $\mathbb{E}(X_1^2 \mid X_1 + X_2 = n)$ is the second moment of a Binomial(n, p) random variable:

$$\begin{aligned}\mathbb{E}(X_1^2 \mid X_1 + X_2 = n) &= n(n-1)p^2 + np, \\ \mathbb{E}(X_2^2 \mid X_1 + X_2 = n) &= n(n-1)(1-p)^2 + n(1-p).\end{aligned}$$

   By the linearity of conditional expectation,

$$\mathbb{E}(X_1^2 + X_2^2 \mid X_1 + X_2 = n) = 2n^2p^2 - 2n^2p - 2np^2 + n^2 + 2np.$$

Remark. We can compute the second moment of a Binomial(n, p) random variable X as follows:

$$\begin{aligned}
\mathbb{E}(X(X-1)) &= \sum_{k=0}^{n} k(k-1)\cdot \mathbb{P}(X=k) \\
&= \sum_{k=2}^{n} k(k-1)\cdot \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= n(n-1) \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} p^k (1-p)^{n-k} \\
&= n(n-1)p^2 \sum_{k=2}^{n} \binom{n-2}{k-2} p^{k-2} (1-p)^{(n-2)-(k-2)} \\
&= n(n-1)p^2,
\end{aligned}$$

and thus $\mathbb{E}(X^2) = \mathbb{E}(X(X-1)) + \mathbb{E}(X) = n(n-1)p^2 + np$.

2. Let X and Y be independent random variables with distributions Exponential(μ) and Exponential(λ) respectively. Let $U = \max\{X, Y\}$ and $V = \min\{X, Y\}$.

   a. Find $\mathbb{E}(U)$ and $\mathbb{E}(V)$.

   Solution. By the tail-sum formula,

$$\mathbb{E}(V) = \int_0^\infty \mathbb{P}(V > v)\,\mathrm{d}v = \int_0^\infty \mathbb{P}(X > v, Y > v)\,\mathrm{d}v = \int_0^\infty e^{-\mu v} \cdot e^{-\lambda v}\,\mathrm{d}v = \frac{1}{\mu+\lambda}.$$

   By the identity $X + Y = U + V$ and by the linearity of expectation,

$$\mathbb{E}(U) = \mathbb{E}(X) + \mathbb{E}(Y) - \mathbb{E}(V) = \frac{1}{\mu} + \frac{1}{\lambda} - \frac{1}{\mu+\lambda}.$$

   Remark. We actually found that $\mathbb{P}(V > v) = e^{-(\mu+\lambda)v}$ for every $v \geq 0$; in other words, V has distribution Exponential($\mu + \lambda$).

   b. Find $\operatorname{cov}(U, V)$.

   Hint: This requires no integration.

   Solution. Observe that $UV = XY$. Then,

$$\mathbb{E}(UV) = \mathbb{E}(XY) = \mathbb{E}(X) \cdot \mathbb{E}(Y) = \frac{1}{\mu\lambda},$$

$$\operatorname{cov}(U, V) = \mathbb{E}(UV) - \mathbb{E}(U) \cdot \mathbb{E}(V) = \frac{1}{\mu\lambda} - \frac{1}{\mu(\mu+\lambda)} - \frac{1}{\lambda(\mu+\lambda)} + \frac{1}{(\mu+\lambda)^2}.$$

   c. Find the probability density function, $f_Z(z)$, of $Z = V/U$.

   Solution. For $0 < z \leq 1$, we compute

$$f_Z(z) = \int_0^\infty f_{Z,U}(z, u)\,\mathrm{d}u = \int_0^\infty |u| \cdot f_{V,U}(uz, u)\,\mathrm{d}u.$$

   As a reminder, u is the Jacobian determinant of the linear transformation $(v, u) \mapsto (z, u)$:

$$\left|\det\begin{bmatrix} \partial v/\partial z & \partial v/\partial u \\ \partial u/\partial z & \partial u/\partial u \end{bmatrix}\right| = \left|\det\begin{bmatrix} u & z \\ 0 & 1 \end{bmatrix}\right| = |u|.$$

   The joint distribution of V and U is given by, whenever $v < u$,

$$f_{V,U}(v, u) = f_{X,Y}(v, u) + f_{X,Y}(u, v) = f_X(v) f_Y(u) + f_X(u) f_Y(v) = \mu\lambda(e^{-(\mu v + \lambda u)} + e^{-(\mu u + \lambda v)}).$$

   Thus, for $0 < z < 1$,

$$\begin{aligned} f_Z(z) &= \int_0^\infty |u| \cdot \mu\lambda(e^{-(\mu u z + \lambda u)} + e^{-(\mu u + \lambda u z)})\,\mathrm{d}u \\ &= \mu\lambda \int_0^\infty u e^{-(\mu z + \lambda)u} + u e^{-(\mu + \lambda z)u}\,\mathrm{d}u \\ &= \frac{\mu\lambda}{(\mu z + \lambda)^2} + \frac{\mu\lambda}{(\mu + \lambda z)^2}, \end{aligned}$$

   and for $z = 1$,

$$f_Z(1) = \int_0^\infty u \cdot \mu\lambda e^{-(\mu+\lambda)u}\,\mathrm{d}u = \frac{\mu\lambda}{(\mu+\lambda)^2}.$$

3. Let $n \geq 2$, and let $X_1, \ldots, X_n$ be i.i.d. Uniform$([0,1])$ random variables. Let A denote the number of *ascents* in the sequence $(X_1, \ldots, X_n)$:

$$A = \#\{1 \leq i \leq n-1 : X_i < X_{i+1}\}.$$

Similarly, let D denote the number of *descents* in $(X_1, \ldots, X_n)$:

$$D = \#\{1 \leq j \leq n-1 : X_j > X_{j+1}\}.$$

Note: Your answers to the following questions should be functions of n.

Hint: This problem requires no integration at all.

a. Find $\mathbb{P}(A = 0)$, and find $\mathbb{E}(A)$.

Solution. By the symmetry of all $n!$ possible orderings of $X_1, \ldots, X_n$,

$$\mathbb{P}(A = 0) = \mathbb{P}(X_1 > \cdots > X_n) = \frac{1}{n!}.$$

Since the given random variables are i.i.d. and continuous, we know that with probability one, $X_1, \ldots, X_n$ take on n distinct values. Hence, with probability one, $A + D = n - 1$. Symmetry again implies

$$\mathbb{E}(A) = \frac{n-1}{2}.$$

Remark. One way to see the second instance of symmetry is to consider $1 - X_1, \ldots, 1 - X_n$.

b. Find $\mathbb{P}(A = 1 \mid X_1 < X_2)$.

Solution. If $n = 2$, then $\mathbb{P}(A = 1 \mid X_1 < X_2) = 1$. Otherwise, $n \geq 3$, and

$$\begin{aligned}
\mathbb{P}(A = 1 \mid X_1 < X_2) &= \mathbb{P}(X_2 > \cdots > X_n \mid X_1 < X_2) \\
&= \frac{\mathbb{P}(X_1 < X_2, X_2 > \cdots > X_n)}{\mathbb{P}(X_1 < X_2)} \\
&= \frac{(n-1)/n!}{1/2} \\
&= \frac{2(n-1)}{n!}.
\end{aligned}$$

(It turns out that this formula holds for all $n \geq 2$.)

c. Find $\mathbb{P}(X_i < X_{i+1}, X_j > X_{j+1})$ for all i and j.

Solution. If $i = j$, then $\mathbb{P}(X_i < X_{i+1}, X_j > X_{j+1}) = 0$. Otherwise, take $i < j$ without loss of generality. In the particular case where $i + 1 = j$,

$$\mathbb{P}(X_i < X_{i+1}, X_j > X_{j+1}) = \mathbb{P}(X_{j-1} < X_j, X_j > X_{j+1}) = \frac{2}{3!} = \frac{1}{3}.$$

In the remaining case where $i + 1 < j$, we have by independence that

$$\mathbb{P}(X_i < X_{i+1}, X_j > X_{j+1}) = \mathbb{P}(X_i < X_{i+1}) \cdot \mathbb{P}(X_j > X_{j+1}) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

In summary, for $1 \leq i, j \leq n-1$,

$$\mathbb{P}(X_i < X_{i+1}, X_j > X_{j+1}) = \begin{cases} 0 & \text{if } i = j, \\ 1/3 & \text{if } i + 1 = j \text{ or } i - 1 = j, \\ 1/4 & \text{otherwise.} \end{cases}$$

d. Find $\operatorname{cov}(A, D)$.

Solution. Write A and D as sums of $n-1$ indicator functions. By the bilinearity of covariance,

$$\begin{aligned}\operatorname{cov}(A, D) &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \operatorname{cov}(\mathbb{1}\{X_i < X_{i+1}\}, \mathbb{1}\{X_j > X_{j+1}\}) \\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \mathbb{P}(X_i < X_{i+1}, X_j > X_{j+1}) - \mathbb{P}(X_i < X_{i+1}) \cdot \mathbb{P}(X_j > X_{j+1}),\end{aligned}$$

where $n-1$ terms correspond to the case of $i = j$ and $2(n-2)$ terms correspond to the case of $i+1 = j$ or $i-1 = j$. Therefore, by part (c),

$$\operatorname{cov}(A, D) = (n-1)\left(0 - \frac{1}{4}\right) + 2(n-2)\left(\frac{1}{3} - \frac{1}{4}\right) = -\frac{n}{12} - \frac{1}{12}.$$