Qualifying Exam: Applied Probability

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Answer all three questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem, you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

1. a. Let X_1, X_2, X_3 be independent random variables with distribution Exponential(1). Find

$$\mathbb{E}\left(\frac{X_1}{X_1 + X_2 + X_3}\right).$$

Solution. By symmetry, $\mathbb{E}(X_1/(X_1 + X_2 + X_3)) = 1/3$.

In more detail, this follows from the linearity of expectation and the fact that X_1, X_2, and X_3 are i.i.d.:

$$1 = \mathbb{E}\left(\frac{X_1 + X_2 + X_3}{X_1 + X_2 + X_3}\right) = \mathbb{E}\left(\frac{X_1}{X_1 + X_2 + X_3}\right) + \mathbb{E}\left(\frac{X_2}{X_1 + X_2 + X_3}\right) + \mathbb{E}\left(\frac{X_3}{X_1 + X_2 + X_3}\right) = 3\,\mathbb{E}\left(\frac{X_1}{X_1 + X_2 + X_3}\right).$$

b. Let X and Y be independent random variables with distribution Uniform($[0,1]$), and let $V = X + Y$.

Find the joint probability density function of X and V; find the conditional probability density function of X given $V = v$; and find $\mathbb{E}(X \mid V)$.

Solution. By direct computation,

$$\begin{aligned} f_{X,V}(x,v) &= f_{X,Y}(x, v-x) \\ &= f_X(x) \cdot f_Y(v-x) \\ &= \mathbb{1}\{x \in [0,1] \text{ and } v \in [x, x+1]\}; \end{aligned}$$

$$\begin{aligned} f_V(v) &= \int_0^1 f_{X,V}(x,v)\,\mathrm{d}x \\ &= \int_0^1 \mathbb{1}\{v \in [x, x+1]\}\,\mathrm{d}x \\ &= \begin{cases} v & \text{if } 0 \le v \le 1, \\ 2 - v & \text{if } 1 \le v \le 2; \end{cases} \end{aligned}$$

$$\begin{aligned} f_{X \mid V}(x \mid v) &= \frac{f_{X,V}(x,v)}{f_V(v)} \\ &= \begin{cases} \frac{1}{v} & \text{if } 0 \le x \le v \le 1 \text{ and } 0 < v, \\ \frac{1}{2-v} & \text{if } 1 \le v \le x+1 \le 2 \text{ and } v < 2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now, rather than calculating $\mathbb{E}(X \mid V)$ using $f_{X|V}(x \mid v)$, it suffices to observe by symmetry that

$$\mathbb{E}(X \mid V) = \frac{V}{2}.$$

In more detail, this follows from the linearity of conditional expectation and the fact that X and Y are i.i.d., as in part (a):

$$1 = \mathbb{E}(X+Y \mid X+Y) = \mathbb{E}(X \mid X+Y) + \mathbb{E}(Y \mid X+Y) = 2\,\mathbb{E}(X \mid X+Y).$$

2. Suppose that in an election, candidate A receives n votes, and candidate B receives m votes, where $n > m$. Note that there are $\binom{n+m}{n}$ possible orders in which the $n+m$ votes are counted. Assuming that all $\binom{n+m}{n}$ such orderings are equally likely, show that the probability that candidate A is always ahead in the count of votes is $(n-m)/(n+m)$.

   Solution. Let E be the event that candidate A is always ahead in the count of votes, and observe that because $n > m$, the complement of E is the event that there is a tie at some point during the count of votes. Let F be the event that the first counted vote is for candidate A. Then,

$$\mathbb{P}(E^c) = \mathbb{P}(E^c \cap F) + \mathbb{P}(E^c \cap F^c).$$

   We now make two observations:

   i. $\mathbb{P}(E^c \cap F) = \mathbb{P}(E^c \cap F^c)$. For each ordering of votes in which the event $E^c \cap F$ occurs, we can find a unique corresponding ordering in which $E^c \cap F^c$ occurs: replace each vote that is counted before the first tie with its opposite vote. For example, the ordering corresponding to $(A, A, B, B; B, A, A)$ is $(B, B, A, A; B, A, A)$.

   ii. $E^c \cap F^c = F^c$. If the first counted vote is for candidate B, then, because $n > m$, there must be a tie at some point during the count of votes.

   Thus, $\mathbb{P}(E^c) = 2\,\mathbb{P}(F^c) = 2m/(n+m)$, and

$$\mathbb{P}(E) = 1 - \frac{2m}{n+m} = \frac{n-m}{n+m}.$$

3. Let n be a positive integer with prime factorization $n = p_1^{m_1} \cdots p_k^{m_k}$, where $p_1, \ldots, p_k$ are distinct primes and $m_1, \ldots, m_k \geq 1$. Let N be an integer chosen uniformly at random from the set $\{1, \ldots, n\}$. Show that

$$\mathbb{P}(N \text{ shares no prime factor in common with } n) = \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$

Solution. For each $i \in \{1, \ldots, k\}$, let A_i be the event that N is an integer multiple of p_i. Then, by the principle of inclusion-exclusion,

$$\begin{aligned}
\mathbb{P}(N \text{ shares no prime factor in common with } n) &= 1 - \mathbb{P}(A_1 \cup \cdots \cup A_k) \\
&= 1 - \sum_{j=1}^{k} (-1)^{j-1} \sum_{n_1 < \cdots < n_j} \mathbb{P}(A_{n_1} \cap \cdots \cap A_{n_j}) \\
&= 1 - \sum_{j=1}^{k} (-1)^{j-1} \sum_{n_1 < \cdots < n_j} \frac{1}{p_{n_1} \cdots p_{n_j}} \\
&= 1 + \sum_{j=1}^{k} \sum_{n_1 < \cdots < n_j} \left(-\frac{1}{p_{n_1}}\right) \cdots \left(-\frac{1}{p_{n_j}}\right) \\
&= \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right).
\end{aligned}$$