Qualifying Exam: Applied Probability

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Answer all three questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem, you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

1. We say that a permutation π has i as a *fixed point* if $\pi(i) = i$.

   a. For each $n \geq 1$, find the probability p_n that a random permutation on $\{1, \ldots, n\}$ has no fixed points.

   Hint: Consider the principle of inclusion-exclusion.

   Solution. For each $i \in \{1, \ldots, n\}$, let A_i be the event that i is a fixed point of the given random permutation. The principle of inclusion-exclusion yields

$$\begin{aligned} p_n &= 1 - \mathbb{P}(A_1 \cup \cdots \cup A_n) \\ &= 1 - \sum_{i=1}^{n} (-1)^{i-1} \sum_{k_1 < \cdots < k_i} \mathbb{P}(A_{k_1} \cap \cdots \cap A_{k_i}) \\ &= 1 - \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \frac{(n-i)!}{n!} \\ &= \sum_{i=0}^{n} (-1)^i \frac{1}{i!}. \end{aligned}$$

   b. Let S be a subset of $\{1, \ldots, n\}$ with size k. Find the probability that S is equal to the set of fixed points of a random permutation on $\{1, \ldots, n\}$, and find the probability that such a random permutation has exactly k fixed points.

   Solution. Observe that S is equal to the set of fixed points of a permutation π if and only if π fixes every point in S and π has no fixed points as a permutation on $\{1, \ldots, n\} \setminus S$. Hence, by part (a),

$$\mathbb{P}(\text{the set of fixed points of a random permutation is equal to } S) = \frac{(n-k)!}{n!} p_{n-k}.$$

   The sum of the above probability over all $\binom{n}{k}$ distinct choices of S is

$$\mathbb{P}(\text{a random permutation has exactly } k \text{ fixed points}) = \binom{n}{k} \frac{(n-k)!}{n!} p_{n-k} = \frac{p_{n-k}}{k!}.$$

   c. Show that as n tends to infinity, the distribution of the number of fixed points of a random permutation converges to Poisson(1).

Solution. For each $k \in \mathbb{N}$,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(\text{a random permutation on } \{1,\ldots,n\} \text{ has exactly } k \text{ fixed points}) &= \lim_{n\to\infty} \frac{p_{n-k}}{k!} \\
&= \frac{1}{k!} \lim_{n\to\infty} \sum_{i=0}^{n-k} (-1)^i \frac{1}{i!} \\
&= \frac{1}{k!} e^{-1},
\end{aligned}$$

which is indeed the probability that a Poisson(1)-distributed random variable is equal to k.

2. Let $(S_n)_{n\geq 0}$ be a simple symmetric random walk that starts from 0 on the integers, let $T = \min\{n \geq 1 : S_n = 0\}$, and write $\mathbb{P}_a(\cdot) = \mathbb{P}(\cdot \mid S_0 = a)$.

   a. For $a \geq 1$, $i \geq 1$, and $n \geq 1$, express $\mathbb{P}_a(S_n = i, T \leq n)$ and $\mathbb{P}_a(S_n = i, T > n)$ in terms of finitely many basic probabilities, which are probabilities of the form $\mathbb{P}_0(S_n = k)$, $\mathbb{P}_0(S_n \leq k)$, or $\mathbb{P}_0(S_n \geq k)$.

      Hint: Consider the reflection principle.

      Solution. By reflecting about 0 the random walk's path starting from the first visit to 0, we find that

$$\begin{aligned}
\mathbb{P}_a(S_n = i, T \leq n) &= \mathbb{P}_a(S_n = i, T < n) \\
&= \mathbb{P}_a(S_n = -i, T < n) \\
&= \mathbb{P}_a(S_n = -i) \\
&= \mathbb{P}_0(S_n = -i - a) \\
&= \mathbb{P}_0(S_n = i + a).
\end{aligned}$$

      It follows that

$$\begin{aligned}
\mathbb{P}_a(S_n = i, T > n) &= \mathbb{P}_a(S_n = i) - \mathbb{P}_a(S_n = i, T \leq n) \\
&= \mathbb{P}_0(S_n = i - a) - \mathbb{P}_0(S_n = i + a).
\end{aligned}$$

   b. For $a \geq 1$ and $n \geq 1$, show that

$$\mathbb{P}_a(T > n) = \sum_{j=1-a}^{a} \mathbb{P}_0(S_n = j).$$

      Hint: Use part (a), and look for cancellation.

      Solution. By the law of total probability and by part (a),

$$\begin{aligned}
\mathbb{P}_a(T > n) &= \sum_{i=1}^{a+n} \mathbb{P}_a(S_n = i, T > n) \\
&= \sum_{i=1}^{a+n} \mathbb{P}_0(S_n = i - a) - \sum_{i=1}^{a+n} \mathbb{P}_0(S_n = i + a) \\
&= \sum_{j=1-a}^{n} \mathbb{P}_0(S_n = j) - \sum_{j=1+a}^{n} \mathbb{P}_0(S_n = j) \\
&= \sum_{j=1-a}^{a} \mathbb{P}_0(S_n = j).
\end{aligned}$$

   c. You may use the fact that for each $j \in \mathbb{Z}$,

$$\lim_{m\to\infty} \frac{\mathbb{P}_0(S_{2m} = 2j)}{1/\sqrt{\pi m}} = 1.$$

      For $a \geq 1$, find c and β such that

$$\lim_{n\to\infty} \frac{\mathbb{P}_a(T > n)}{c/n^{\beta}} = 1.$$

      Does c or β depend on a?

      Hint: It suffices to consider even n.

      Solution. Assume that n is even, write $n = 2m$, and observe that $\mathbb{P}_0(S_n = i) \neq 0$ only if i is even. By part (b) and by the given fact, we find that as m tends to infinity,

$$\mathbb{P}_a(T > 2m) = \sum_{i=1-a}^{a} \mathbb{P}_0(S_{2m} = i) \sim \frac{a}{\sqrt{\pi m}}.$$

In other words, taking $c = a\sqrt{2/\pi}$ and $\beta = 1/2$, we have that

$$\lim_{n\to\infty} \frac{\mathbb{P}_a(T > n)}{c/n^{\beta}} = 1.$$

Note that c depends on a, whereas β does not.

Above, the limit as n tends to infinity exists by the observation that if n is odd, then

$$\left(\frac{n-1}{n}\right)^{\beta} \cdot \frac{\mathbb{P}_a(T > n-1)}{c/(n-1)^{\beta}} \geq \frac{\mathbb{P}_a(T > n)}{c/n^{\beta}} \geq \left(\frac{n+1}{n}\right)^{\beta} \cdot \frac{\mathbb{P}_a(T > n+1)}{c/(n+1)^{\beta}},$$

where $n-1$ and $n+1$ are even.

3. Let X and Y be independent random variables with distribution Normal$(0,1)$.

   a. Find $a \in \mathbb{R}$ for which $U = X + 2Y$ and $V = aX + Y$ are independent.

   Solution. We know that (U,V) has multivariate normal distribution, so U and V are independent if and only if $\operatorname{cov}(U,V) = 0$. From the computation

$$\begin{aligned} \operatorname{cov}(U,V) &= \operatorname{cov}(X,aX) + \cancel{\operatorname{cov}(X,Y)} + \cancel{\operatorname{cov}(2Y,aX)} + \operatorname{cov}(2Y,Y) \\ &= a\operatorname{var}(X) + 2\operatorname{var}(Y) \\ &= a + 2, \end{aligned}$$

   it follows that U and V are independent when $a = -2$.

   b. For every $b \in \mathbb{R}$, find $\mathbb{E}(XY \mid X + 2Y = b)$.

   Hint: Use part (a).

   Solution. Observe that we can write X and Y as linear combinations of U and V, namely $X = (U - 2V)/5$ and $Y = (2U + V)/5$. By the independence of U and V, we find that

$$\begin{aligned} \mathbb{E}(XY \mid X + 2Y = b) &= \frac{\mathbb{E}((U-2V)(2U+V) \mid U = b)}{25} \\ &= \frac{2b^2 - 3b\mathbb{E}(V) - 2\mathbb{E}(V^2)}{25} \\ &= \frac{2b^2 - \cancel{3b\mathbb{E}(V)} - 2\operatorname{var}(V)}{25} \\ &= \frac{2}{25}b^2 - \frac{2}{5}. \end{aligned}$$