Spring 2015, Applied Probability:

1. For any n, we know that $\sum_{k=1}^{n} X_k \sim \text{Poisson}\left(\sum_{k=1}^{n} \lambda_k\right)$. Let $\varepsilon > 0$ be arbitrary and fixed. Consider

$$P\left(\left|\frac{\sum_{k=1}^{n} X_k}{\sum_{k=1}^{n} \lambda_k} - 1\right| \geq \varepsilon\right) = P\left(\left|\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} \lambda_k\right| \geq \varepsilon \sum_{k=1}^{n} \lambda_k\right)$$

$$\leq \frac{E\left[\left|\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} \lambda_k\right|^2\right]}{\varepsilon^2 \left(\sum_{k=1}^{n} \lambda_k\right)^2}$$

$$= \frac{\text{Var}\left(\sum_{k=1}^{n} X_k\right)}{\varepsilon^2 \left(\sum_{k=1}^{n} \lambda_k\right)^2}$$

$$= \frac{\left(\sum_{k=1}^{n} \lambda_k\right)}{\varepsilon^2 \left(\sum_{k=1}^{n} \lambda_k\right)^2}$$

$$= \frac{1}{\varepsilon^2 \sum_{k=1}^{n} \lambda_k}$$

Taking limit as $n \to \infty$, since $\sum_{k=1}^{\infty} \lambda_k = \infty$, we obtain

$$\lim_{n \to \infty} P\left(\left|\frac{\sum_{k=1}^{n} X_k}{\sum_{k=1}^{n} \lambda_k} - 1\right| \geq \varepsilon\right) = 0$$

2. Define $X = \sum_{i=1}^{51} \mathbb{1}_{A_i}$ where A_i is the event that i^{th} and $(i+1)^{st}$ are the same

a) $E[X] = \sum_{i=1}^{51} P(A_i)$

$$P(A_i) = \frac{\binom{4}{2}}{\binom{52}{2}} * 13 = \frac{4 \cdot 3}{\cancel{2}} \; \frac{\cancel{2}}{\cancel{52} * 51} * \cancel{13} = \frac{1}{17}$$

So, $E[X] = \sum_{i=1}^{51} \frac{1}{17} = 3$

b) $\text{Var}(X) = \sum_{i=1}^{51} \text{Var}(\mathbb{1}_{A_i}) + 2 \sum_{1 \leq i < j \leq 51} \text{Cov}(\mathbb{1}_{A_i}, \mathbb{1}_{A_j})$

$\rightarrow \text{Var}(\mathbb{1}_{A_i}) = P(A_i) - P(A_i)^2 = \frac{1}{17}\left(1 - \frac{1}{17}\right) = \frac{16}{289}$

→ If $j = i+1$: $\quad Cov(1_{A_i}, 1_{A_j}) = P(A_i \cap A_j) - P(A_i)P(A_j)$

$$= \frac{\binom{4}{3}}{\binom{52}{3}} \times 13 - \frac{1}{289}$$

$$= \frac{4 \times 3 \times 2}{\cancel{52} \times \cancel{51} \times \cancel{50}}_{\;17 \quad 25} \times \cancel{13} - \frac{1}{289}$$

$$= \frac{-8}{17^2 * 25}$$

→ If $j \geq i+2$: $\quad Cov(1_{A_i}, 1_{A_j}) = P(A_i \cap A_j) - P(A_i)P(A_j)$

$$= P(A_j | A_i)P(A_i) - P(A_i)P(A_j)$$

$$= \left(\frac{\binom{2}{2}}{\binom{50}{2}} + \frac{\binom{4}{2}}{\binom{50}{2}} * 12 \right) \frac{1}{17} - \frac{1}{289}$$

$$= \left(\frac{\cancel{2}}{\cancel{50} * 49}_{25} + \frac{6 * \cancel{2} * 12}{50 * 49}_{25} \right) * \frac{1}{17} - \frac{1}{289}$$

$$= \frac{73}{5 * 49} * \frac{1}{17} - \frac{1}{289}$$

$$= \frac{1241}{5 * 49 * (17)^2} - \frac{1}{(17)^2}$$

$$= \frac{996}{5 * 49 * 17^2}$$

So,

$$Var(X) = \frac{\cancel{51}^{\,3} \times 16}{\cancel{289}_{17}} + 2 * \cancel{50}^{\,2} * \frac{-8}{17^2 * \cancel{25}} + 2 * 49 * \cancel{25}^{\,5} * \frac{996}{\cancel{5} * \cancel{49} * (17)^2}$$

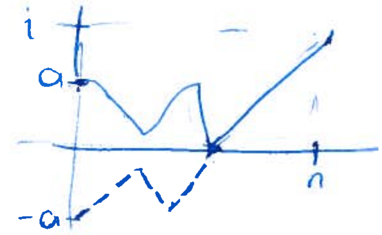
$$= \frac{48}{17} - \frac{32}{17^2} + \frac{9960}{17^2}$$

$$= \frac{10744}{289}$$

c) $P(X = 39)$ is the probability that all of the numbers $(A, 2, 3, \dots, J, Q, K)$ are grouped togetger. So,

$$P(X = 39) = \frac{13! \; (4!)^{13}}{52!}$$

$T = \min\{n > 0 : S_n = 0\}$



$\mathbb{P}_a(S_n = i, T \leq n)$ for $a > 0$

$= \mathbb{P}_{-a}(S_n = i)$

$P_a(S_n \geq i, T > n)$ $\frac{a>0}{i,n\geq 0}$

$\|$

a 'da başlayıp i'de bitenler

— $T_n \leq n$ = -a da başlayıp i'de bitenler

$\cup$

— $T_n > n$

$\mathbb{P}_a(S_n \geq i) = 1$

$P_a(S_n \geq i, T > n) = P_a(S_n \geq i) - P_a(S_n \geq i, T_n \leq n)$

$= P_a(S_n \geq i) - P_{-a}(S_n \geq i)$

$= \sum_{k=i}^{\infty} P_a(S_n = k) - \sum_{k=i}^{\infty} P_a(S_n = k)$

$= \sum_{k=i}^{\infty} [P_a($

West Kolye 0219
232 883

$\{$

$n, k-a$

$\binom{n}{m}\left(\frac{1}{2}\right)^n$

$m = \frac{n+k-a}{2}$

$= \sum_{k=i}^{\infty} [P_a(S_n = k) - P_a(S_n = k)]$

$\sum_{k=i}^{\infty} \left[\binom{n}{\frac{n+k-a}{2}} \frac{1}{2^n} - \binom{n}{\frac{n+k+a}{2}} \frac{1}{2^n} \right]$

$= \frac{1}{2^n} \sum_{k=i}^{\infty} \left[\binom{n}{\frac{n-a}{2} + \frac{k}{2}} - \binom{n}{\frac{n+a}{2} + \frac{k}{2}} \right]$

$= \frac{1}{2^n} \sum_{k=i}^{n-a}$

$\frac{n-a}{2} + \frac{k}{2} \leq n$

$n - a + k \leq 2n$

$\boxed{\begin{matrix} k - a \leq n \\ k + a \leq n \end{matrix}}$

$\frac{n+a}{2} + \frac{k}{2} \leq n$

$n + a + k \leq 2n$

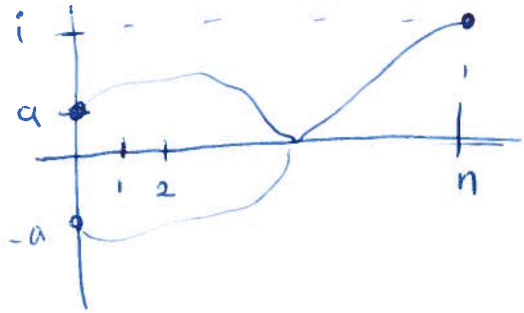
$k + a \leq n$

Call

call

5243.	.1215927
5244.	.1444267
5245.	.1105418
5246.	.3739063
5247.	.2641677
5248.	.2810553
5249.	.1966646
5250.	.0773446
5251.	.0701672
5252.	.2493338
5253.	.2194934
5254.	.240091
5255.	.[illegible]769193
5256.	.4551708
5257.	.0975352
5258.	.0810651
5259.	.2754133
5260.	.2726697
5261.	.1293996
5262.	.2426514
5263.	.2364552
5264.	.3644556
5265.	.2073167
5266.	.3947792
5267.	.4333436
5268.	.1453843
5269.	.2595214
5270.	.2013097
5271.	.1298397
5272.	.2346963
5273.	.1551946
5274.	.3133776
5275.	.1600424
5276.	.1133135
5277.	.2840638
5278.	.3647679
5279.	.5086656
5280.	.1250206
5281.	.2810553
5282.	.288702
5283.	.1510098
5284.	.2244971
5285.	.3612386
5286.	.2698276
5287.	.278609
5288.	.4752137
5289.	.3030246
5290.	.1182314
5291.	.3513374
5292.	.3284246
5293.	.3244716
5294.	.2455692
5295.	.1133135
5296.	.2738777
5297.	.3495123
5298.	.3996417
5299.	.0791149

5300.	.2442482
5301.	.2219872
5302.	.3132033
5303.	.2603506
5304.	.1314668
5305.	.2308419
5306.	.2630866
5307.	.300154
5308.	.3360558
5309.	.0925613
5310.	.4885461
5311.	.3513374
5312.	.2072682
5313.	.215783
5314.	.37079
5315.	.1541944
5316.	.1215927
5317.	.1054445
5318.	.2715307
5319.	.2727768
5320.	.334524
5321.	.1729676
5322.	.1149363
5323.	.2617168
5324.	.3056369
5325.	.2133297
5326.	.1276354
5327.	.2282921
5328.	.2953997
5329.	.2795044
5330.	.2629815
5331.	.3834981
5332.	.2411987
5333.	.1620966
5334.	.318004
5335.	.2853231
5336.	.2097293
5337.	.3147545
5338.	.3541918
5339.	.1506119
5340.	.1062131
5341.	.3071122
5342.	.1370819
5343.	.125888
5344.	.2084723
5345.	.2411987
5346.	.2232402
5347.	.3085905
5348.	.233819
5349.	.2232402
5350.	.28692
5351.	.2826408
5352.	.3090636
5353.	.2060682
5354.	.305967
5355.	.2881854



$$\mathbb{P}_{-a}(S_n = i) =$$

$$\mathbb{P}_a(S_n = i + 2a)$$

$$\sum_{k=i}^{\infty} P_a(S_n = k) - \sum_{k=i}^{\infty} P_a(S_n = k + 2a)$$

$$\sum_{k=i}^{\infty} P_0(S_n = k - a) - \sum_{k=1}^{\infty} P_0(S_n = k + a)$$

$$\sum_{k=i}^{\infty} \left[P_0(S_n = k - a) - P_0(S_n = k + a) \right] = P_0(S_n = i - a) - P_0(S_n = i + a) \quad k = i$$

$$P_0(S_n = i+1-a) - P_0(S_n = i+1+a) \quad k = i+1$$

$$P_0(S_n = i+2-a) - P_0(S_n = i+2+a) \quad k = i+2$$

$$P_0(S_n = i+3-a) - P_0(S_n = i+3+a) \quad k = i+3$$

$\vdots$

$$P_0(S_n = i) \qquad k =$$

$k = i$ $\quad P_a(S_n = i) - P_a(S_n = i + 2a)$

$k = i+1$ $\quad P_a(S_n = i+1) - P_a(S_n = i + 2a + 1)$

$k = i+2$ $\quad P_a(S_n = i+2) - P_a(S_n = i + 2a + 2)$

$k = i+2a-1$ $\quad P(S_n = i+2a-1) - P(S_n = i + 4a - 1)$

$k = i+2a$ $\quad P(S_n = i+2a) - P_a(S_n = i + 4a)$

$$\sum_{k=i}^{i+2a-1} P_a(S_n = k) = P(S_n \geq i, T > n)$$

k

$$\frac{n+a}{2} \leq n$$

$$a \leq n$$

$$P_a(T > n) = P_a(S_n > 0, T > n) = P_a(S_n \geq 1, T > n) = \sum_{k=1}^{2a} P_a(S_n = k)$$

$$= \sum_{k=1}^{2a} \binom{n}{\frac{n+k-a}{2}} \frac{1}{2^n} = \sum_{k=1}^{2a} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi \frac{n+k-a}{2}} \left(\frac{n+k-a}{e}\right)^{\frac{n+k-a}{2}}}$$