Spring 2014, Applied Probability:

1. Define $R = \sum_{i=1}^{m-2} 1_{A_i}$ where A_i is the event that the player wins i^{th} and $(i-1)^{st}$ games.

i) $\mathbb{E}[R] = \sum_{i=1}^{m-1} \mathbb{E}[1_{A_i}] = \sum_{i=1}^{m-1} P(A_i) = \sum_{i=1}^{m-1} a_i a_{i+1}$.

$$var(R) = \sum_{i=1}^{m-1} var(1_{A_i}) + 2 \sum_{1 \le i < j \le m}^{m-1} cov(1_{A_i}, 1_{A_j})$$

$\longrightarrow$ $var(1_{A_i}) = \mathbb{E}[1_{A_i}^2] - (\mathbb{E}[1_{A_i}])^2 = P(A_i) - P(A_i)^2 = a_i a_{i+1}(1 - a_i a_{i+1})$.

$\longrightarrow$ $\underline{j = i+1:}$ $cov(1_{A_i}, 1_{A_j}) = \mathbb{E}[1_{A_i} 1_{A_j}] - \mathbb{E}[1_{A_i}]\mathbb{E}[1_{A_j}]$

$$= a_i a_{i+1} a_{i+2} - a_i a_{i+1}^2 a_{i+2}$$

$$= a_i a_{i+1} a_{i+2}(1 - a_{i+1})$$

$\longrightarrow$ $\underline{j \ge i+2:}$ $cov(1_{A_i}, 1_{A_j}) = 0$ because of independence.

So,

$$Var(R) = \sum_{i=1}^{m-1} a_i a_{i+1}(1 - a_i a_{i+1}) + 2 \sum_{i=1}^{m-2} a_i a_{i+1} a_{i+2}(1 - a_{i+1})$$

ii) $m = 100$, $a_n = .1$ $\forall n$. Then,

$$\mathbb{E}[R] = \sum_{i=1}^{99} 0.01 = 0.99$$

$$Var(R) = \sum_{i=1}^{99} 0.01 * 0.99 + 2 \sum_{i=1}^{98} 0.001 * 0.9$$

$$= \frac{(99)^2}{10000} + \frac{98 * 18}{10000}$$

$$= 1.1565$$

2. $X_1, \ldots, X_n$ are iid with pdf $f(x) = \frac{x}{2}$ for $0 < x < 2$.

a) $F_S(s) = P(S \le s) = P(X_1 + X_2 \le s)$

For $\underline{0 < s \le 2:}$

$$P(X_1 + X_2 \le s) = \int_0^s \int_0^{s-x_2} \frac{x_1 x_2}{4} dx_1 dx_2 = \frac{1}{4} \int_0^s x_2 \int_0^{s-x_2} x_1 dx_1 dx_2 = \frac{1}{4} \int_0^s \frac{x_2 (s-x_2)^2}{2} dx_2$$

$$= \frac{1}{8}\int_0^s (x_2^3 - 2sx_2^2 + s^2x_2)\,dx_2 = \frac{1}{8}\left(\frac{s^4}{4} - \frac{2s^4}{3} + \frac{s^4}{2}\right) = \frac{s^4}{96}$$

For $\underline{2 < s \leq 4}$:

$$P(X_1+X_2 \leq s) = \int_0^2 \int_0^{s-2} \frac{x_1x_2}{4}\,dx_1dx_2 + \int_{s-2}^{2}\int_0^{s-x_1} \frac{x_1x_2}{4}\,dx_2dx_1$$

$$= \frac{1}{4}\int_0^2 x_2 \frac{(s-2)^2}{2}\,dx_2 + \frac{1}{4}\int_{s-2}^{2} x_1 \frac{(s-x_1)^2}{2}\,dx_1$$

$$= \frac{(s-2)^2}{4} + \frac{1}{8}\int_{s-2}^{2}(x_1^3 - 2sx_1^2 + s^2x_1)\,dx_1$$

$$= \frac{(s-2)^2}{4} + \frac{1}{8}\left[\frac{x_1^4}{4} - \frac{2sx_1^3}{3} + \frac{s^2x_1^2}{2}\right]_{s-2}^{2}$$

$$= \frac{(s-2)^2}{4} + \frac{1}{8}\left[4 - \frac{16s}{3} + 2s^2 - \frac{(s-2)^4}{4} + \frac{2s(s-2)^3}{3} - \frac{s^2(s-2)^2}{2}\right]$$

$$= \frac{(s-2)^2}{4} + \frac{1}{2} - \frac{2s}{3} + \frac{s^2}{4} - \frac{(s-2)^4}{32} + \frac{s(s-2)^3}{12} - \frac{s^2(s-2)^2}{16}$$

$$= \frac{(s-2)^2}{4}\left[1 - \frac{1}{8} + \frac{s(s-2)}{3} - \frac{s^2}{4}\right] + \frac{s^2}{4} - \frac{2s}{3} + \frac{1}{2}$$

$$= \frac{(s-2)^2}{4}\left[\frac{s^2}{12} - \frac{2s}{3} + \frac{7}{8}\right] + \frac{s^2}{4} - \frac{2s}{3} + \frac{1}{2}$$

So,

$$F_S(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ \frac{s^2}{96} & \text{if } 0 < s \leq 2 \\ \frac{(s-2)^2}{4}\left(\frac{s^2}{12} - \frac{2s}{3} + \frac{7}{8}\right) + \frac{s^2}{4} - \frac{2s}{3} + \frac{1}{2} & \text{if } 2 \leq s < 4 \\ 1 & \text{if } s \geq 4 \end{cases}$$

(probably a mistake in integration provide another solution at the end)

b) $F_L(\ell) = P(L \leq \ell)$

$$= P(\min(X_1, \ldots, X_{100}) \leq \ell)$$

$$= 1 - P(\min(X_1, \ldots, X_{100}) > \ell)$$

$$= 1 - P(X_1 > \ell, \ldots, X_{100} > \ell)$$

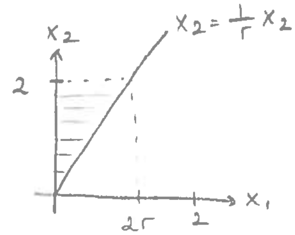
$$= 1 - \prod_{i=1}^{n} P(X_i > \ell)$$

$$= 1 - \prod_{i=1}^{n} \int_\ell^2 \frac{x}{2}\,dx$$

$$= 1 - \left(1 - \frac{\ell^2}{4}\right)^n, \quad 0 < \ell < 2.$$

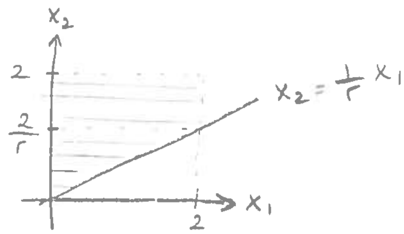
c) $F_R(r) = P\left(\frac{X_1}{X_2} \le r\right) = P(X_1 \le r X_2)$, $0 < r < \infty$

→ If $0 < r < 1$:



$$F_R(r) = \int_0^2 \int_0^{r x_2} \frac{x_1 x_2}{4} \, dx_1 dx_2 = \frac{1}{4} \int_0^2 x_2 \frac{r^2 x_2^2}{2} dx_2 = \frac{r^2}{8} \frac{x_2^4}{4} \Big|_0^2 = \frac{r^2}{2}$$

→ If $r \geq 1$:



$$F_R(r) = \int_0^2 \int_{\frac{x_1}{r}}^2 \frac{x_1 x_2}{4} \, dx_2 dx_1 = \frac{1}{4} \int_0^2 x_1 \left(2 - \frac{x_1^2}{2r^2}\right) dx_1 = \frac{1}{4} \left[x_1^2 - \frac{x_1^4}{8r^2} \right]_0^2$$

$$= \frac{1}{4}\left(4 - \frac{2}{r^2}\right) = 1 - \frac{1}{2r^2}$$

Thus,

$$F_R(r) = \begin{cases} 0 & \text{if } r \leq 0 \\ \frac{r^2}{2} & \text{if } 0 < r < 1 \\ 1 - \frac{1}{2r^2} & \text{if } 1 \leq r < \infty \end{cases}$$

d) We know that

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} [F_X(x)]^{j-1} f_X(x) [1 - F_X(x)]^{n-j}$$

So, consider for $0 < x < 2$

$$F_X(x) = \int_0^x \frac{t}{2} dt = \frac{x^2}{4}$$

Thus,

$$f_{X_{(10)}}(x) = \frac{100!}{9!\,90!} \left(\frac{x^2}{4}\right)^9 \frac{x}{2} \left(1 - \frac{x^2}{4}\right)^{90}$$

$$= \frac{100!}{9!\,90!} \left(\frac{x}{2}\right)^{19} \left(1 - \frac{x^2}{4}\right)^{90}$$

* Consider finding pdf of X_1+X_2 for part (a) by using convolution:

$$f_S(s) = \int_{-\infty}^{\infty} f_{X_1}(x_1)\, f_{X_2}(s-x_1)\,dx_1 = \int_0^2 f_{X_1}(x_1)\, f_{X_2}(s-x_1)\,dx_1 \quad , \quad 0<s<4$$

$\rightarrow$ If $0<s<2$. $$f_S(s) = \int_0^s \frac{x_1}{2}\cdot\frac{s-x_1}{2}\,dx_1 = \frac{1}{4}\left[\frac{sx_1^2}{2} - \frac{x_1^3}{3}\right]_0^s = \frac{s^3}{24}$$

$\rightarrow$ If $2\le s<4$. $$f_S(s) = \int_{s-2}^{s} \frac{x_1}{2}\,\frac{s-x_1}{2}\,dx_1 = \frac{1}{4}\left(2s - \frac{8}{3} - \frac{s(s-2)^2}{2} - \frac{(s-2)^3}{3}\right)$$

So,

$$f_S(s) = \begin{cases} \frac{s^3}{24} & \text{if } 0<s<2 \\ \frac{1}{4}\left(2s-\frac{8}{3}-\frac{s(s-2)^2}{2}-\frac{(s-2)^3}{8}\right) & \text{if } 2\le s<4 \\ 0 & \text{otherwise} \end{cases}$$

3. Consider that $\mathbb{E}\left[\frac{S_n}{n}\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \mu$ Now, by using Chebychev's Inequality we have

$$P\left(\left|\frac{S_n}{n}-\mu\right| \geqslant \varepsilon\right) \le \frac{1}{\varepsilon^2}\,\mathbb{E}\left[\left|\frac{S_n}{n}-\mu\right|^2\right] = \frac{1}{\varepsilon^2}\operatorname{var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2\varepsilon^2}\operatorname{var}(S_n)$$

Now, observe that

$$\operatorname{Var}(S_n) = \sum_{i=1}^{n}\operatorname{Var}(X_i) + 2\underbrace{\sum_{1\le i<j\le n}\operatorname{Cov}(X_i,X_j)}_{=0\ \forall i\neq j} = \sum_{i=1}^{n}\operatorname{Var}(X_i) \le Cn$$

So, we get

$$P\left(\left|\frac{S_n}{n}-\mu\right| \geqslant \varepsilon\right) \le \frac{1}{n^2\varepsilon^2}Cn = \frac{C}{n\varepsilon^2}$$

which is the best upper bound I obtain for the probability. Taking limit as $n\to\infty$, we get

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n}-\mu\right| \geqslant \varepsilon\right) = 0.$$