1b 3a, c, e

Math 505a 2013 Spring Qualifying Exam

1. a) Let X and Y be square integrable random variables such that

$$E(X|Y) = Y \quad \text{and} \quad E(Y|X) = X. \tag{1}$$

Show that

$$P(X = Y) = 1. \tag{2}$$

b) Prove that (1) implies (2) under the weakened assumption that X and Y are integrable.

2. Suppose k balls are tossed into n boxes, with all n^k possibilities equally likely. Let D be the number of boxes that contain exactly 2 balls.

a) Compute $p := P($ exactly 2 balls land in box 1$)$.

b) In terms of p, give an exact expression for the mean ED.

c) Compute $r := P($ exactly 2 balls land in box 1 and exactly 2 balls land in box 2$)$.

d) Give an exact expression for the second moment ED^2 in terms of p and r.

e) Compute the variance of D.

3. a) Suppose $g(u) := Eu^S$ is the probability generating function of a non-negative integer valued random variable S satisfying $P(S > 0) > 0$. Let T be distributed as S, conditional on the event $S > 0$. Express $h(u) := Eu^T$, the probability generating function of T, in terms of $g(u)$.

T integer valued? nonnegative?

In parts b) and c) below, N is a nonnegative integer valued random variable with probability generating function $f(u) := Eu^N$, and S is the number of heads in N tosses of a $p \in (0,1)$ coin, with all coin tosses having probability p of coming up heads, independently of each other and of N.

$T|S>0 \sim S$

$E[u^T] = E[u^T \mid S>0]P\{S>0\} + E[u^T \mid S=0]\, P\{S=0\}$

$= g(u)\, P\{S>0\} + \dfrac{E[u^T 1_{\{S=0\}}]}{P\{S=0\}}\, P\{S=0\}$

$= g(u)\, P\{S>0\} + E[u^T 1_{\{S=0\}}]$

b) Write the probability generating function $g(u) := Eu^S$ of S in a simple form.

c) Now combine parts a) and b): what is the probability generating function h of the number T of heads, in N tosses of a p-coin, conditional on getting at least one head, when N has probability generating function f?
Parts d,e) can be worked on even if you are stumped by a,b,c).

$h(u) = \mathbb{E}[u^T]$ is this the same as T in this question

? $\mathbb{E}[u^T \mid T>0]$

d) Suppose someone claims that for $\alpha \in (0,1)$, the function

$$f(u) := 1 - (1-u)^\alpha$$

is a probability generating function of a nonnegative, non constant integer valued random variable N. What properties of f must you check? Is the hypothesis $\alpha > 0$ used? What happens in the cases $\alpha = 0, \alpha = 1$ and $\alpha > 1$?

e) Combine parts a)-d), that is suppose $\alpha \in (0,1)$, N has the generating function $f(u) := 1 - (1-u)^\alpha$, and T is the number of heads in N tosses of a p-coin, conditional on getting at least one head. Do N and T have the same distribution?

$$P(N=k) = \frac{f^{(k)}(0)}{k!} \Rightarrow \quad P(N=0) = 0$$

$$P(N=1) = \alpha$$

$$P(N=k) = \frac{1}{k!}\prod_{i=0}^{k-1}(\alpha - i), \quad k \geq 1$$

- "$\alpha > 0$" is requiered since $P(N=1) = \alpha$
- If $\alpha = 0$ $f(u)$ is not a prob. g. func.
- If $\alpha = 1$ $P(N=1) = 1$
- If $\alpha > 1$ $f(u)$ is not a prob. g func
  since $P(N=1) = \alpha$

Spring 2013, Applied Probability

1\. a) $\hookrightarrow$ $\underline{Var(X|Y)=0}$:

$$Var(X) = \mathbb{E}[Var(X|Y)] + Var(\mathbb{E}[X|Y])$$
$$= \mathbb{E}[Var(X|Y)] + Var(Y)$$
$$= \mathbb{E}[Var(X|Y)] + \mathbb{E}[Var(Y|X)] + Var(\mathbb{E}[Y|X])$$
$$= \mathbb{E}[Var(X|Y) + Var(Y|X)] + Var(X)$$

Since $Var(X) < \infty$ (because of assumption) we get $\mathbb{E}[\underbrace{Var(X|Y)+Var(Y|X)}_{\geqslant 0}] = 0$

$\Rightarrow Var(X|Y) + Var(Y|X) \geqslant 0 \Rightarrow Var(X|Y) = 0$ and $Var(Y|X) = 0$.

$\longrightarrow$ $\underline{Var(X) = Var(Y)}$:

$$0 = Var(X|Y) = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2 = \mathbb{E}[X^2|Y] - Y^2$$

So, we have $\mathbb{E}[X^2|Y] = Y^2$ and taking expectation of both sides, we obtain $E[X^2] = E[Y^2]$. This also gives us $Var(X) = Var(Y)$ since $\mathbb{E}[X] = \mathbb{E}[Y]$ by assumption $(\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[Y])$.

$\longrightarrow$ $\underline{\mathbb{E}[XY] = \mathbb{E}[X^2]}$:

$$\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY|X]] = \mathbb{E}[X\,\mathbb{E}[Y|X]] = \mathbb{E}[X^2]$$

$\longrightarrow$ $\underline{P(X=Y)=1}$:

$$Var(X-Y) = Var(X) + Var(Y) - 2Cov(X,Y)$$
$$= 2Var(X) - 2\mathbb{E}[XY] + 2\mathbb{E}[X]\,\mathbb{E}[Y]$$
$$= 2\mathbb{E}[X^2] - 2(\mathbb{E}[X])^2 - 2\mathbb{E}[X^2] + 2(\mathbb{E}[X])^2$$
$$= 0$$

So, $P(X-Y=c) = 1$ i.e. $X = Y + c$ a.e. Since $\mathbb{E}[X] = \mathbb{E}[Y]$, we must have $c = 0$.
Thus, $P(X=Y) = 1$

2\. $D = \#$ of boxes that contain exactly 2 balls

a) $X = \#$ of balls in box 1, and $P(\text{any ball lands in box } 1) = \frac{1}{n}$. So, $X \sim \text{binomial}(k, \frac{1}{n})$ and $p = P(X=2) = \binom{k}{2}\left(\frac{1}{n}\right)^2\left(\frac{n-1}{n}\right)^{k-2} = \binom{k}{2} \cdot \frac{(n-1)^{k-2}}{n^k}$.

b) $D = \sum_{i=1}^{n} 1_{D_i}$ where D_i is the event that i^{th} box contains exactly 2 balls. Then, $\mathbb{E}[D] = \sum_{i=1}^{n} P(D_i)$ where $P(D_i) = \binom{k}{2} \frac{(n-1)^{k-2}}{n^k}$ by part (a).

So,

$$\mathbb{E}[D] = \sum_{i=1}^{n} \binom{k}{2} \frac{(n-1)^{k-2}}{n^k} = \sum_{i=1}^{n} p = np$$

c) Firstly, let $X = \#$ of balls land in first box. So, $X \sim \text{Binomial}(k, \frac{1}{n})$, then letting $Y = \#$ of balls land in second box, we see that $Y|X \sim \text{Binomial}(k-X, \frac{1}{n-1})$. So,

$$P(X=2, Y=2) = P(Y=2 \mid X=2)\, P(X=2)$$

$$= \binom{k-2}{2} \frac{1}{(n-1)^2} \left(\frac{n-2}{n-1}\right)^{k-4} \binom{k}{2} \frac{1}{n^2} \left(\frac{n-1}{n}\right)^{k-2}$$

$$= \binom{k}{2}\binom{k-2}{2} \frac{(n-2)^{k-4}}{n^k}$$

(*choose 2 then choose 2 * $\frac{\text{all possible in}}{\text{all possible wi}}$

So, $r = \binom{k}{2}\binom{k-2}{2} \frac{(n-2)^{k-4}}{n^k}$

d) $$\mathbb{E}[D^2] = \sum_{i=1}^{n} \mathbb{E}[1_{D_i}^2] + \sum_{1 \le i < j \le n} \mathbb{E}[1_{D_i} 1_{D_j}]$$

$$= \sum_{i=1}^{n} P(D_i) + \sum_{1 \le i < j \le n} P(D_i \cap D_j)$$

$$= \sum_{i=1}^{n} p + \sum_{1 \le i < j \le n} r$$

$$= np + \frac{(n-1)n}{2} r$$

e) $\text{Var}(D) = \mathbb{E}[D^2] - \mathbb{E}[D]^2 = np + \frac{(n-1)n}{2} r - n^2p^2 = np(1-np) + \frac{(n-1)n}{2} r$

3. a) $$\mathbb{E}[u^T] = \mathbb{E}[u^T \mid S>0]\, P\{S>0\} + \mathbb{E}[u^T \mid S=0]\, P\{S=0\}$$

$$= g(u)\, P\{S>0\} + \frac{\mathbb{E}[u^T 1_{\{S=0\}}]}{P\{S=0\}} P\{S=0\}$$

$$= g(u)\, P\{S>0\} + \mathbb{E}[u^T 1_{\{S=0\}}].$$

b) We have $S \sim \text{Binomial}(N, p)$. Then,

$$\mathbb{E}[u^S \mid N=n] = \sum_{k=0}^{n} u^k P(S=k \mid N=n)$$
$$= \sum_{k=0}^{n} u^k \binom{n}{k} p^k (1-p)^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (pu)^k (1-p)^{n-k}$$
$$= (1-p+pu)^n$$

So, $\mathbb{E}[u^S \mid N] = (1-p+pu)^N$. Then,

$$\mathbb{E}[u^S] = \mathbb{E}\big[\mathbb{E}[u^S \mid N]\big] = \mathbb{E}[(1-p+pu)^N] = f(1-p+pu).$$

c) T = # of heads in N tosses of a p-coin, conditional on getting at least one head

$$m(u) = \mathbb{E}[u^T \mid T>0] = \frac{\mathbb{E}[u^T 1_{\{T>0\}}]}{P\{T>0\}}$$ where $T \sim \text{Binomial}(N, p)$.

$\to$ $P\{T>0\} = \sum_{n=1}^{\infty} P\{T>0 \mid N=n\} P\{N=n\} = \sum_{n=1}^{\infty} [1-(1-p)^n] P\{N=n\} = \underbrace{\sum_{n=1}^{\infty} P\{N=n\}}_{=1<\infty} - \underbrace{\sum_{n=1}^{\infty} (1-p)^n P\{N=n\}}_{<\infty}$

since

$$= 1 - \mathbb{E}[(1-p)^N] = 1 - f(1-p).$$

$\to$ $\mathbb{E}[u^T 1_{\{T>0\}}] = \sum_{k=0}^{\infty} u^k 1_{\{T>0\}} P\{T=k\} = \sum_{k=1}^{\infty} u^k P\{T=k\} = \sum_{k=0}^{\infty} u^k P\{T=k\} - P\{T=0\}$

$$= h(u) - P\{T=0\} = h(u) - 1 + P\{T>0\} = h(u) - 1 + 1 - f(1-p)$$
$$= h(u) - f(1-p)$$

Thus, $m(u) = \dfrac{h(u) - f(1-p)}{1 - f(1-p)}$.

d) To have $f(u) = 1-(1-u)^\alpha$ as a probability generating function of N, we must have

i) $f(1) = 1$ since $f(u) = \mathbb{E}[u^N] = \sum_{k=0}^{\infty} u^k P\{N=k\}$ implies $f(1)=1$. For the given function $f(u)$, $f(1) = 1-(1-1)^\alpha = 1$, if $\alpha \geq 0$, this condition is satisfied.

Note that if $\alpha = 0$, then $f(u) \equiv 1$ which is not a probability generating function. So, let $\alpha > 0$.

ii) $P\{N=k\} = \dfrac{f^{(k)}(0)}{k!}$, i.e. f must be infinitely many times differentiable and $0 < \dfrac{f^{(k)}(0)}{k!} < 1$ and $\sum_{k=0}^{\infty} \dfrac{f^{(k)}(0)}{k!} = 1$.

The given function f is infinitely many times differentiable. Then, consider

$f'(u) = \alpha(1-u)^{\alpha-1}$ $\longrightarrow P\{N=1\} = f'(0) = \alpha$

$f''(u) = -\alpha(\alpha-1)(1-u)^{\alpha-2}$ $\longrightarrow P\{N=2\} = \dfrac{\alpha(1-\alpha)}{2}$

$f'''(u) = \alpha(\alpha-1)(\alpha-2)(1-u)^{\alpha-3}$ $\longrightarrow P\{N=3\} = \dfrac{\alpha(\alpha-1)(\alpha-3)}{6}$

$\vdots$

$f^{(k)}(u) = (-1)^{k-1}\prod_{i=1}^{k-1}(\alpha-i)(1-u)^{\alpha-k}$ $\longrightarrow P\{N=k\} = \dfrac{(-1)^{k-1}}{k!}\prod_{i=1}^{k-1}(\alpha-i)$

Since $P\{N=1\}$ we must have $\alpha \le 1$

If $\alpha = 1$, $P\{N=0\} = 0$ and $P\{N=1\} = 1$

So, we must have $0 < \alpha \le 1$ to have $f(u)$ as a probability generating function

If $\alpha > 1$, $f(u)$ cannot be a probability generating function