Spring 2012, Applied Probability

1. <u>1st way</u>: We can observe that

$$P(M=0) = \frac{1}{101} \quad \text{(prob. of the last person's place is empty)}$$

$$P(M=1) = \frac{1}{101} \quad \text{(prob of "the place of the person who is sitting at the place of the last one is empty")}$$

In the same way we can say that

$$P(M=k) = \frac{1}{101} \quad \text{(prob. of "having the seat of } k^{th} \text{ person that is going to change place as empty")}$$

Thus, we have $P(M=k) = \frac{1}{101}$ For all $k = 0, 1, \ldots, 100$ Then,

$$E[M] = \sum_{k=0}^{100} k\, P(M=k) = \frac{1}{101} \sum_{k=1}^{100} k = \frac{1}{101} * \frac{100 * 101}{2} = 50$$

<u>2nd way</u>: Let M_n denote the number of "changes" when there are n seats and n tickets are sold. Then,

$$E[M_n] = E[M_n \mid n^{th} \text{ person's place is empty}]\, P(n^{th} \text{ person's place is empty})$$
$$+ E[M_n \mid n^{th} \text{ person's place is occupied}]\, P(n^{th} \text{ person's place is occupied})$$
$$= 0 + (E[M_{n-1}] + 1) \cdot \frac{n-1}{n}$$
$$= \frac{n-1}{n} + \frac{n-1}{n} E[M_{n-1}]$$

So,
$$E[M_{101}] = \frac{100}{101} + \frac{100}{101} E[M_{100}]$$
$$= \frac{100}{101} + \frac{99}{101} + \frac{99}{101} E[M_{99}]$$
$$\vdots$$
$$= \frac{100}{101} + \frac{99}{101} + \frac{98}{101} + \cdots + \frac{1}{101} \qquad E[M_1] = 1$$
$$= \frac{1}{101} \sum_{k=1}^{100} k$$
$$= 50$$

2. a) $$E[S] = \sum_{1 \le i < j \le n} E[1_{\{X_i = X_j\}}] = \sum_{1 \le i < j \le n} P\{X_i = X_j\}$$

$$\rightarrow P\{X_i = X_j\} = \sum_{k=1}^{\infty} P\{X_i = X_j = k\} = \sum_{k=1}^{\infty} P\{X_i = k\} P\{X_j = k\} = \sum_{k=1}^{\infty} p_k^2 = f_2$$

So, $E[S] = \sum_{1 \le i < j \le n} f_2 = \binom{n}{2} f_2$.

b) $E[T] = \sum_{1 \le i < j < k \le n} E[1_{\{X_i = X_j = X_k\}}] = \sum_{1 \le i < j < k \le n} P\{X_i = X_j = X_k\}$

$\longrightarrow P\{X_i = X_j = X_k\} = \sum_{m=1}^{\infty} P\{X_i = X_j = X_k = m\} = \sum_{m=1}^{\infty} P\{X_i = m\} P\{X_j = m\} P\{X_k = m\}$

$= \sum_{m=1}^{\infty} P_m^3 = f_3$

So, $E[T] = \sum_{1 \le i < j < k \le n} f_3 = \binom{n}{3} f_3$

c) $S^2 = \sum_{1 \le i < j \le n} 1^2_{\{X_i = X_j\}} + 2 \sum_{1 \le i < j < k \le n} 1_{\{X_i = X_j\}} 1_{\{X_j = X_k\}} + 2 \sum_{1 \le i < j < k < l \le n} 1_{\{X_i = X_j\}} 1_{\{X_k = X_l\}}$

$E[S^2] = \sum_{1 \le i < j \le n} E[1_{\{X_i = X_j\}}] + 2 \sum_{1 \le i < j < k \le n} E[1_{\{X_i = X_j = X_k\}}] + 2 \sum_{1 \le i < j < k < l \le n} E[1_{\{X_i = X_j = X_k = X_l\}}]$

$= \binom{n}{2} f_2 + 2 \binom{n}{3} f_3 + 2 \binom{n}{4} f_4$

$Var(S) = \binom{n}{2} f_2 + 2\binom{n}{3} f_3 + 2\binom{n}{4} f_4 - \binom{n}{2}^2 f_2^2$

3. a) Firstly, we know that since $X \sim exponential(1)$, $X \sim gamma(1,1)$ where the pdf of $Y \sim gamma(\alpha, \beta)$ is

$$f_Y(y) = \frac{x^{\alpha-1} e^{-y/\beta}}{\Gamma(\alpha) \beta^{\alpha}}, y > 0$$

Then, we know that if $Y_1 \sim gamma(\alpha_1, \beta)$ and $Y_2 \sim gamma(\alpha_2, \beta)$ are independent, then $Y_1 + Y_2 \sim gamma(\alpha_1 + \alpha_2, \beta)$. Since X_1, X_2, X_3, X_4 are independent we have $S_4 = X_1 + X_2 + X_3 + X_4 \sim gamma(4,1)$. So, S_4 has pdf,

$$f_{S_4}(t) = \frac{t^3 e^{-t}}{\Gamma(4) 1^4} = \frac{t^3 e^{-t}}{3!}, \quad t > 0$$

b) We know that for any iid r.v. $X_1, \dots, X_n$,

$$f_{X_{(1)}, \dots, X_{(n)}}(x_1, \dots, x_n) = \begin{cases} n! \, f_{X_1}(x_1) \cdots f_{X_n}(x_n) & , -\infty < x_1 < \cdots < x_n < \infty \\ 0 & , \text{otherwise} \end{cases}$$

Since $U_1, U_2, U_3 \sim uniform(0,1)$, we know $f_{U_i}(u) = 1$, $0 < u < 1$ for $i = 1,2,3$.

So, we get the pdf $g(x,y,z)$ of $(U_{(1)}, U_{(2)}, U_{(3)})$ as

$$g(x,y,z) = \begin{cases} 3! & , 0<x<y,z<1 \\ 0 & , \text{otherwise} \end{cases}$$

c) We have X_1, X_2, X_3, X_4 and $S_n = \sum_{i=1}^{n} X_i$ for $n=1,2,3,4$. Now, let us find the pdf of S_1, S_2, S_3, S_4.

Noting $(J)_{ij} = \frac{\partial S_i}{\partial X_j}$, we have

$$J = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1$$

Since $x_1, x_2, x_3, x_4 > 0$ we have $0 < s_1 < s_2 < s_4$ and, since $f_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4) = e^{-(x_1+x_2+x_3+x_4)}$, we obtain that

$$f_{S_1,S_2,S_3,S_4}(s_1,s_2,s_3,s_4) = e^{-s_4} \qquad , 0<s_1<s_2<s_3<s_4$$

Now, let $T_1 = \frac{S_1}{S_4}$, $T_2 = \frac{S_2}{S_4}$, $T_3 = \frac{S_3}{S_4}$ and $T_4 = S_4$ and let us find the pdf of T_1, T_2, T_3, T_4. Again noting that $(J) = \frac{\partial T_i}{\partial S_j}$, we have

$$J = \begin{vmatrix} 1/s_4 & 0 & 0 & -s_1/s_4^2 \\ 0 & 1/s_4 & 0 & -s_2/s_4^2 \\ 0 & 0 & 1/s_4 & -s_3/s_4^2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{s_4^3}$$

Since $0<s_1<s_2<s_3<s_4$, we have $0<t_1t_4<t_2t_4<t_3t_4<t_4$. Since $t_4>0$, we have $0<t_1<t_2<t_3<1$ So,

$$f_{T_1,T_2,T_3,T_4}(t_1,t_2,t_3,t_4) = e^{-t_4} \cdot t_4^3 \quad , \qquad 0<t_1<t_2<t_3<1,\ t_4>0$$

$$= \frac{e^{-t_4}\, t_4^3}{6} * 6, \qquad 0<t_1<t_2<t_3<1\ ,\ t_4>0$$

We know $f_{T_4}(t_4) = \frac{e^{-t_4}\, t_4^3}{6}$, $t_4>0$. So, we deduce that $(T_1,T_2,T_3) = \left(\frac{S_1}{S_4}, \frac{S_2}{S_4}, \frac{S_3}{S_4}\right)$ and $T_4 = S_4$ are independent and

$$f_{T_1,T_2,T_3}(t_1,t_2,t_3) = 6, \qquad 0<t_1<t_2<t_3<1$$

Thus, we conclude that $\left(\frac{S_1}{S_4}, \frac{S_2}{S_4}, \frac{S_3}{S_4}\right) \stackrel{d}{=} (U_{(1)}, U_{(2)}, U_{(3)})$.

d) The desired probability can be expressed as

$$P(U_1 > U_2 + U_3) + P(U_2 > U_1 + U_3) + P(U_3 > U_1 + U_2)$$

and this probability is the same as $3P(U_1 > U_2 + U_3)$, by symmetry.

Now, let us find pdf of $U_2 + U_3$

$$f_{U_2+U_3}(t) = \int_0^1 f_{U_1}(x) f_{U_2}(t-x)\, dx = \int_0^1 f_{U_2}(t-x)\, dx$$

since $f_{U_2}(x) = 1$ when $0<x<1$. We know $0<t<2$ and $f_{U_2}(t-x) = 1$ only when $0<t-x<1 \Leftrightarrow t-1<x<t$.

→ <u>If $0<t<1$:</u> $\quad f_{U_2+U_3}(t) = \int_0^t dx = t$

→ <u>If $1 \leq t < 2$:</u> $\quad f_{U_2+U_3}(t) = \int_{t-1}^1 dx = 2-t$

So,

$$f_{U_1+U_2}(t) = \begin{cases} t & \text{if } 0<t<1 \\ 2-t & \text{if } 1\leq t<2 \\ 0 & \text{otherwise} \end{cases}$$

Now, consider

$$P(U_1 > U_2 + U_3) = \int_0^1 \int_t^1 f_{U_1}(u) f_{U_2+U_3}(t)\, du\, dt$$

Since we have $t<u<1$ and $0<t<1$, we must have $0<t<u<1$ and also $f_{U_1}(u) = 1$, $f_{U_2+U_3}(t) = t$ in the above integral So,

$$P(U_1 > U_2 + U_3) = \int_0^1 \int_t^1 t\, du\, dt = \int_0^1 (t - t^2)\, dt = \left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Thus, the desired probability is $3 * \frac{1}{6} = \frac{1}{2}$.