MATH 505a **QUALIFYING EXAM** **January 31, 2012**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1.) One hundred and one passengers bought tickets on a 101-seat carriage. One seat was reserved for each passenger. The first 100 passengers took the seats at random so that all 101! possible seating arrangements (with one empty seat) are equally likely. The last passenger insisted on taking the assigned seat. If that seat is occupied, then the passenger in that seat has to move to the corresponding assigned seat, and so on. Compute the expected value of the number M of passengers who have to change their seats. [HINTS: one method is to use a recursion in n, for n in the role of 101, for the expectation, without knowing the distribution of M. Another method is to find the distribution of M explicitly.]

2.) Suppose $\mathbb{P}(X = k) = p_k$ and $p_1 + p_2 + \cdots = 1$. Suppose that $X, X_1, X_2, \ldots, X_n$ are independent and identically distributed. Let $S = \sum_{1\le i<j\le n} 1(X_i = X_j)$ be the number of matching (unordered) *pairs*, and let $T = \sum_{1\le i<j<k\le n} 1(X_i = X_j = X_k)$ be the number of matching (unordered) *trios*. For $r = 1, 2, 3, \ldots,$, let $f_r = \sum p_i^r$, so that $f_1 = 1$.
a) Give a simple expression for $\mathbb{E}\, S$ in terms of n, f_2.
b) Give a simple expression for $\mathbb{E}\, T$ in terms of n, f_3.
c) Give a simple expression for $\operatorname{Var} \mathrm{S}$ in terms of n, f_2, f_3, f_4.

3.) Let X, X_1, X_2, X_3, X_4 be independent standard exponentially distributed random variables, so that $\mathbb{P}(X > x) = e^{-x}$ for $x > 0$. Write $S_n = X_1 + \cdots + X_n$. The goal is to show that the triple $(S_1/S_4, S_2/S_4, S_3/S_4)$ is distributed like the order statistics of three independent standard uniform (0,1) random variables.
a) Give a reason why the density of S_4 is $f(t) = t^3 e^{-t}/6$ for $t > 0$. You may either quote the density for the Gamma family in general, or you may argue about the time of the fourth arrival in a standard, rate 1 Poisson process, or you may carry out the four-fold convolution!
b) With (U_1, U_2, U_3) distributed uniformly in the cube $(0,1)^3$, and $U_{[i]}$ defined to be the ith smallest of U_1, U_2, U_3, show why the density of $(U_{[1]}, U_{[2]}, U_{[3]})$ is $g(x, y, z) = 6$, on the region $0 < x < y < z < 1$.
c) Show, with detail, why the triple $(S_1/S_4, S_2/S_4, S_3/S_4)$ is distributed like the order statistics of three independent standard uniform (0,1) random variables, AND that this triple is independent of S_4. This should include calculation of both a 4 by 4 Jacobian matrix, and calculation of its determinant, also known as the “Jacobian”.
d) Consider three independent uniform $(0, 1)$ variables. Compute the probability that the largest exceeds the sum of the other two.