Spring 2010, Applied Probability.

1 Since (X_1, X_2) has standard bivariate distribution, we know that given $X_1 = x$, X_2 has $N\left(0+\frac{3}{5}\cdot\frac{1}{1}(x-0), \left(1-\left(\frac{3}{5}\right)^2\right)1\right) = N\left(\frac{3x}{5}, \frac{16}{25}\right)$ distribution.

a) $E[Y_2 \mid Y_1 = 90] = E[75+2X_2 \mid 80+3X_1 = 90] = 75 + 2E[X_2 \mid X_1 = \frac{10}{3}]$. Since, given $X_1 = \frac{10}{3}$, $X_2 \sim N\left(2, \frac{16}{25}\right)$, we deduce that $E[Y_2 \mid Y_1 = 90] = 79$.

b) $P(Y_2 > 75 \mid Y_1 = 90) = P(75+2X_2 > 75 \mid 80+3X_1 = 90)$

$$= P\left(X_2 > 0 \mid X_1 = \tfrac{10}{3}\right)$$

$$= P(X > 0) \quad \text{where} \quad X \sim N\left(2, \tfrac{16}{25}\right)$$

$$= P\left(\frac{X-2}{4/5} > -\frac{2}{4/5}\right)$$

$$= P(Z > -2.5)$$

$$\approx 0.9938$$

2. $$E[e^{tS_{\tau_n}} \mid \tau_n = k] = E[e^{tS_k} \mid \tau_n = k]$$

$$= E\left[e^{t\sum_{i=1}^{k} X_i} \mid \tau_n = k\right]$$

by independence

$$= \prod_{i=1}^{k} E[e^{tX_i}]$$

$$= \left(\frac{e^t + e^{-t}}{2}\right)^k$$

So, $E[e^{tS_{\tau_n}} \mid \tau_n] = \left(\frac{e^t+e^{-t}}{2}\right)^{\tau_n}$. Now, we need to find the probability generating function of τ_n.

Let us call the probability generating function of τ_n by $G(s)$

So,

$$M_{S_{\tau_n}}(t) = G\left(\frac{e^t + e^{-t}}{2}\right) = G(\cosh t)$$

$$M'_{S_{\tau_n}}(t) = G'(\cosh t)\sinh t$$

$\rightarrow$ $E[S_{\tau_n}] = M'_{S_{\tau_n}}(0) = G'(1) * 0 = 0$

$$M''_{S_{\tau_n}}(t) = G''(\cosh t)(\sinh t)^2 + G'(\cosh t)\cosh t$$

$\rightarrow$ $E[S^2_{\tau_n}] = M''_{S_{\tau_n}}(0) = G'(1) * 1 = E[\tau_n]$.

$\rightarrow$ $Var(S_{\tau_n}) = E[S^2_{\tau_n}] - (E[S_{\tau_n}])^2 = E[\tau_n]$

3. a) Suppose we have the random sample $\begin{bmatrix}\cos\theta_1\\ \sin\theta_1\end{bmatrix}, \begin{bmatrix}\cos\theta_2\\ \sin\theta_2\end{bmatrix}, \ldots, \begin{bmatrix}\cos\theta_k\\ \sin\theta_k\end{bmatrix}, \ldots$

$\rightarrow$ $E[\cos\theta_i] = \int_0^{2\pi} \cos\theta \frac{1}{2\pi} d\theta = 0$, $E[\sin\theta_i] = \int_0^{2\pi} \sin\theta \frac{1}{2\pi} d\theta = 0$

$\rightarrow$ $E[\cos^2\theta_i] = \int_0^{2\pi} \cos^2\theta \frac{1}{2\pi} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{4\pi}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{1}{2}$

$\rightarrow$ $E[\sin^2\theta_i] = \int_0^{2\pi} \sin^2\theta \frac{1}{2\pi} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta = \frac{1}{4\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{1}{2}$

$\rightarrow$ $Var(\cos\theta_i) = \frac{1}{2}$, $Var(\sin\theta_i) = \frac{1}{2}$

$\rightarrow$ $E[\cos\theta_i \sin\theta_i] = \int_0^{2\pi} \cos\theta\sin\theta \frac{1}{2\pi} d\theta = \frac{1}{2\pi}\left[\frac{\sin^2\theta}{2}\right]_0^{2\pi} = 0$

Now, let us choose $a_n = \frac{\sqrt{2}}{n}$ and define $A_k = \sqrt{2}\cos\theta_k$ and $B_k = \sqrt{2}\sin\theta_k$ Then, we get

$$E[A_k] = E[B_k] = 0, \quad Var(A_k) = Var(B_k) = 1, \quad Cov(A_k, B_k) = 0$$

Then, by using multivariate CLT,

$$\sqrt{n}\begin{bmatrix} a_n X_n \\ b_n Y_n \end{bmatrix} = \sqrt{n}\begin{bmatrix} \frac{\sqrt{2}}{n} X_n \\ \frac{\sqrt{2}}{n} Y_n \end{bmatrix} = \sqrt{n}\begin{bmatrix} \frac{1}{n}\sum_{k=1}^{n} A_k \\ \frac{1}{n}\sum_{k=1}^{n} B_k \end{bmatrix} \xrightarrow{d} N(0, I)$$

Thus, if we choose $a_n = \frac{\sqrt{2}}{n}$, $(a_n X_n, a_n Y_n)$ has limiting bivariate normal distribution ...

b) By part (a), we know that $\sqrt{\frac{2}{n}}X_n \xrightarrow{d} N(0,1)$ and $\sqrt{\frac{2}{n}}Y_n \xrightarrow{d} N(0,1)$ as $n\to\infty$ Since $g(x)=x^2$ is a continuous function, $\frac{2}{n}X_n^2 \xrightarrow{d} \chi^2_1$ and $\frac{2}{n}Y_n^2 \xrightarrow{d} \chi^2_1$ as $n\to\infty$ Then by independence, $\frac{2}{n}(X_n^2+Y_n^2) \xrightarrow{d} \chi^2_2$ as $n\to\infty$ We know $\chi^2_2 = \Gamma(1,2) = \text{exponential}(2)$

So, $\frac{2}{n}(X_n^2+Y_n^2) \xrightarrow{d} T$ where $T\sim \text{exponential}(2)$

Now, observe that

$$P(R_n^2 \geq n) = P(X_n^2+Y_n^2 \geq n) = P\left(\tfrac{2}{n}(X_n^2+Y_n^2) \geq 2\right) = 1-P\left(\tfrac{2}{n}(X_n^2+Y_n^2)<2\right)$$

$$\Rightarrow \lim_{n\to\infty} P(R_n^2 \geq n) = 1-\lim_{n\to\infty} P\left(\tfrac{2}{n}(X_n^2+Y_n^2)<2\right) = 1-P(T<2) = 1-\int_0^2 \frac{1}{2}e^{-x/2}dx$$

$$= 1-\left[-e^{-x/2}\right]_0^2 = 1-(-e^{-1}+1) = e^{-1}$$

Note for question #2: Define $A_k = n-S_k$ and $B_k = n+S_k$. Then

$$\tau_n = \inf\{k\geq 1 \quad S_k \notin (-n,n)\} = \inf\{k\geq 1 \quad A_k B_k = 0\}$$

Now, we want to find $E[\tau_n]$. Firstly consider that $A_n B_n + n$ is a martingale

$$\mathbb{E}[A_{n+1}B_{n+1}+n+1 \mid A_n, B_n] = \frac{(A_n+1)(B_n-1)}{2} + \frac{(A_n-1)(B_n+1)}{2} + n+1$$

$$= \frac{A_nB_n - \cancel{A_n} + \cancel{B_n} - 1 + A_nB_n + \cancel{A_n} - \cancel{B_n} - 1}{2} + n+1$$

$$= \frac{2(A_nB_n-1)}{2} + n+1$$

$$= A_nB_n + n$$

So, by Optional Stopping Theorem, $\mathbb{E}[\tau_n] = \mathbb{E}[A_{\tau_n}B_{\tau_n}+\tau_n] = \mathbb{E}[A_1B_1+1]$

$$\to \mathbb{E}[A_1B_1+1] = n^2+n\,\mathbb{E}[S_1]-n\,\mathbb{E}[S_1]-\mathbb{E}[S_1^2]+1 = n^2-\mathbb{E}[X_1^2]+1 = n^2-1+1 = n^2$$

Thus $\mathbb{E}[\tau_n]=n^2$ and $Var(S_{\tau_n}) = n^2$.