Spring 2009, Applied Probability:

1. a) $P(N=n) = P\{X_1 \geq X_2 \geq X_3 \geq \cdots \geq X_{n-2} \geq X_{n-1},\ X_{n-1} < X_n\}$

$= P\{X_1 > X_2 > X_3 > \cdots > X_{n-2} > X_{n-1},\ X_{n-1} < X_n\}$ since X_i's are continuous r.v.

Since X_i's are iid, any ordering of X_i's is equally likely. That is the events $\{X_{\pi(1)} > X_{\pi(2)} > \cdots > X_{\pi(n)}\}$ are all have the same probability for all permutations since there are $n!$ possibilities, any ordering has probability $\frac{1}{n!}$. On the other hand for the event $\{X_1 > X_2 > \cdots > X_{n-1},\ X_{n-1} < X_n\}$, there are $n-1$ possible places for X_n. Thus, we deduce,

$$P(N=n) = P\{X_1 > X_2 > \cdots > X_{n-1},\ X_{n-1} < X_n\} = \frac{n-1}{n!}$$

b) $P(N \geq n) = P\{X_1 > X_2 > \cdots > X_{n-1}\} = \frac{1}{(n-1)!}$

c) $E[N] = \sum_{n=2}^{\infty} n P\{N=n\} = \sum_{n=2}^{\infty} n \cdot \frac{n-1}{n!} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = e$

2. a) Let A be the event that the first chosen card is red, and let B denote the number of red cards among f cards. Then,

$$P(A) = \sum_{k=0}^{f} P(A \mid B=k) P(B=k)$$

$$= \sum_{k=0}^{f} \frac{c+k}{c+d+f} \frac{\binom{a}{k}\binom{b}{f-k}}{\binom{a+b}{f}}, \quad f-b \leq k \leq a$$

$$= \frac{c}{c+d+f} \underbrace{\sum_{k=0}^{f} \frac{\binom{a}{k}\binom{b}{f-k}}{\binom{a+b}{f}}}_{=1} + \frac{1}{c+d+f} \underbrace{\sum_{k=0}^{f} k \frac{\binom{a}{k}\binom{b}{f-k}}{\binom{a+b}{f}}}_{\substack{= E[X] \text{ where } X \sim \text{Hypergeometric}(a+b,a,f) \\ = \frac{fa}{a+b}}}$$

$$= \frac{1}{c+d+f}\left(c + \frac{af}{a+b}\right)$$

$$= \frac{ac+bc+af}{(a+b)(c+d+f)}$$

b) Let us denote the event that card comes from deck #1 by C. So, we are trying to find $P(C|A) = \dfrac{P(C\cap A)}{P(A)}$

$$P(C\cap A) = \sum_{k=0}^{f} P(C\cap A \mid B=k)\, P(B=k)$$

$$= \sum_{k=0}^{f} \frac{k}{c+d+f} \frac{\binom{a}{k}\binom{b}{f-k}}{\binom{a+b}{f}}, \quad f-b \le k \le a$$

$$= \frac{1}{c+d+f} \cdot \frac{af}{a+b}$$

$$= \frac{af}{(c+d+f)(a+b)}$$

So, $P(C|A) = \dfrac{af}{(c+d+f)(a+b)} \quad \dfrac{(a+b)(c+d+f)}{ac+bc+af} = \dfrac{af}{ac+bc+af}$

c) Now, consider

$$P(C) = \sum_{k=0}^{f} P(C\mid B=k)\, P(B=k)$$

$$= \sum_{k=0}^{f} \frac{f}{c+d+f} \frac{\binom{a}{k}\binom{b}{f-k}}{\binom{a+b}{f}}, \quad f-b \le k \le a$$

$$= \frac{f}{c+d+f}$$

We see that $P(A)P(C) = \dfrac{(ac+bc+af)f}{(a+b)(c+d+f)^2}$, $P(A\cap C) = \dfrac{af}{(c+d+f)(a+b)}$

$$P(A)P(C) = P(A\cap C) \iff \frac{ac+bc+af}{(c+d+f)} = a \iff ac+bc+af = ac+ad+af$$

$$\iff bc = ad$$

So, if $bc = ad$, A and C are independent

3. a) $\mathbb{E}[Z] = \mathbb{E}\left[\mathbb{E}[S_Y \mid Y]\right]$

$$\mathbb{E}[S_Y \mid Y=n] = \sum_{i=1}^{n} \mathbb{E}[X_i \mid Y=n] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n}\left(\int_0^\infty \pi t e^{-\pi t}\, dt\right) = \sum_{i=1}^{n} \frac{1}{\pi} = \frac{n}{\pi}$$

Thus $\mathbb{E}[S_Y \mid Y] = \dfrac{Y}{\pi}$

$$\mathbb{E}[Z] = \frac{1}{\pi}\,\mathbb{E}[Y] = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{k}{2^k} = \frac{2}{\pi}$$ by the note below.

b) $$\mathbb{E}[s^Y] = \sum_{n=1}^{\infty} s^n P\{Y=n\} = \sum_{n=1}^{\infty}\left(\frac{s}{2}\right)^n = \frac{s}{2-s}, \quad |s|<2$$

c) $$\mathbb{E}[e^{\beta X}] = \int_0^{\infty} e^{\beta t}\,\pi e^{-\pi t}\,dt = \pi\int_0^{\infty} e^{(\beta-\pi)t}\,dt = \pi\left[\frac{e^{(\beta-\pi)t}}{\beta-\pi}\right]_0^{\infty} = \frac{\pi}{\beta-\pi}(0-1) = \frac{\pi}{\pi-\beta}, \beta<\pi.$$

d) $$\mathbb{E}[e^{\beta Z}\mid Y=n] = \mathbb{E}[e^{\beta S_n}\mid Y=n] = \mathbb{E}[e^{\beta S_n}] = \prod_{i=1}^{n}\mathbb{E}[e^{\beta X_i}] = \prod_{i=1}^{n}\frac{\pi}{\pi-\beta} = \left(\frac{\pi}{\pi-\beta}\right)^n, \beta<\pi$$

$$\Rightarrow \mathbb{E}[e^{\beta Z}\mid Y] = \left(\frac{\pi}{\pi-\beta}\right)^Y$$

$$\Rightarrow \mathbb{E}[e^{\beta Z}] = \mathbb{E}\left[\left(\frac{\pi}{\pi-\beta}\right)^Y\right] = \frac{\frac{\pi}{\pi-\beta}}{2-\frac{\pi}{\pi-\beta}} = \frac{\pi}{\pi-2\beta}, \quad \left|\frac{\pi}{\pi-\beta}\right|<2,\ \beta<\pi$$

Then $\left|\frac{\pi}{\pi-\beta}\right|<2 \Rightarrow \pi<2\pi-2\beta \Rightarrow 2\beta<\pi \Rightarrow \beta<\frac{\pi}{2}$

So, $\mathbb{E}[e^{\beta Z}] = \frac{\pi}{\pi-2\beta}$, $\beta<\frac{\pi}{2}$.

e) We are allowed to take derivative in expectation above. So, taking derivative wrt β three times,

$$\mathbb{E}[Z^3 e^{\beta Z}] = \frac{d^3}{d\beta^3}\left(\frac{\pi}{\pi-2\beta}\right) = \frac{d^2}{d\beta^2}\left(\frac{2\pi}{(\pi-2\beta)^2}\right) = \frac{d}{d\beta}\left(\frac{8\pi}{(\pi-2\beta)^3}\right) = \frac{48\pi}{(\pi-2\beta)^4}$$

Then, setting $\beta=0$, we get

$$\mathbb{E}[Z^3] = \frac{48}{\pi^3}$$

* Note that from here we find $\mathbb{E}[Ze^{\beta Z}] = \frac{2\pi}{(\pi-2\beta)^2} \Rightarrow \mathbb{E}[Z] = \frac{2}{\pi}$ for part (a)