Spring 2008, Applied Probability

1. $$\mathbb{E}[X 1_{\{Y\geq t\}}] = \int_0^\infty \mathbb{E}[X 1_{\{Y\geq t\}} | Y=y]\, f_Y(y)\,dy$$

$$= \int_0^\infty 1_{\{Y\geq t\}}\, \mathbb{E}[X | Y=y]\, f_Y(y)\,dy$$

$$= \int_t^\infty \mathbb{E}[X|Y=y]\, f_Y(y)\,dy$$

Then,

$$\int_0^\infty \mathbb{E}[X 1_{\{Y\geq t\}}]\,dt = \int_0^\infty \int_t^\infty \mathbb{E}[X|Y=y]\, f_Y(y)\,dy\,dt$$

since X and Y are square-integrable

$$= \int_0^\infty \int_0^y \mathbb{E}[X|Y=y]\, f_Y(y)\,dt\,dy$$

$$= \int_0^\infty y\, \mathbb{E}[X|Y=y]\, f_Y(y)\,dy$$

$$= \int_0^\infty \mathbb{E}[XY|Y=y]\, f_Y(y)\,dy$$

$$= \mathbb{E}\left[\mathbb{E}[XY|Y]\right]$$

$$= \mathbb{E}[XY]$$

2. a) $\mathrm{var}(a\cdot X + b\cdot Y) = \mathrm{var}(aX) + \mathrm{Var}(bY) + \mathrm{Cov}(aX, bY) + \mathrm{Cov}(bY, aX)$

$\to \mathrm{Cov}(aX, bY) = \sum_{i=1}^m \sum_{j=1}^m a_i b_j \,\mathrm{Cov}(X_i, Y_j) = \sum_{i=1}^m a_i * \left(\sum_{j=1}^m b_j C_{ij}\right) = a^T C b$

$\to \mathrm{Cov}(bY, aX) = \sum_{i=1}^m \sum_{j=1}^m b_i a_j \,\mathrm{Cov}(Y_i, X_j) = \sum_{i=1}^m b_i \left(\sum_{j=1}^m a_j C_{ji}\right) = b^T C^T a$

So, $\mathrm{Var}(a\cdot X + b\cdot Y) = a^T \Sigma_X a + b^T \Sigma_Y b + a^T C b + b^T C^T a$.

b) $$k u = \mathrm{var}((a+ub)\cdot X)$$

$$= (a+ub)^T \Sigma_X (a+ub)$$

$$= a^T \Sigma_X a + u\, a^T \Sigma_X b + u\, b^T \Sigma_X a + u^2 b^T \Sigma_X b$$

So, we obtain

$$a^T \Sigma_X a + (a^T \Sigma_X b + b^T \Sigma_X a - k) u + b^T \Sigma_X b u^2 = 0$$

Since this is valid $\forall u \in \mathbb{R}$, we must have

i) $a^T \Sigma_X a = 0 \Rightarrow var(a \cdot X) = 0 \Rightarrow \exists c_1 \in \mathbb{R}$ s.t. $P(a \cdot X = c_1) = 1$.

ii) $b^T \Sigma_X b = 0 \Rightarrow var(b \cdot X) = 0 \Rightarrow \exists c_2 \in \mathbb{R}$ s.t. $P(b \cdot X = c_2) = 1$

iii) $a^T \Sigma_X b + b^T \Sigma_X a - k = 0$.

Since $b^T \Sigma_X a \in \mathbb{R}$, we have $b^T \Sigma_X a = (b^T \Sigma_X a)^T = a^T \Sigma_X b$

So, we have $k = 2 a^T \Sigma_X b$

3. a) For any nonzero $v \in \mathbb{R}$,

$$F_{\frac{1}{U}}(v) = P\left(\frac{1}{U} \le v\right) = P\left(U \ge \frac{1}{v}\right) = 1 - P\left(U < \frac{1}{v}\right) = 1 - F_U\left(\frac{1}{v}\right)$$

Then taking derivatives of both sides wrt v,

$$f_{\frac{1}{U}}(v) = -f_U\left(\frac{1}{v}\right) * \frac{-1}{v^2} = \frac{1}{v^2} \cdot \frac{1}{\pi} \cdot \frac{1}{1 + \frac{1}{v^2}} = \frac{1}{\pi} \cdot \frac{1}{1+v^2}$$

So, pdf of $1/U$ is the same as the pdf of U a.c. Thus, they have the same distribution.

b) Now consider by using Cauchy-Schwarz Inequality,

$$1 = \mathbb{E}\left[|U|^{\alpha/2} \frac{1}{|U|^{\alpha/2}}\right]^2 \le \mathbb{E}[|U|^\alpha] * \mathbb{E}\left[\left|\frac{1}{U}\right|^\alpha\right]$$

Since U and $\frac{1}{U}$ have the same distribution, we have $\mathbb{E}[|U|^\alpha] = \mathbb{E}\left[\left|\frac{1}{U}\right|^\alpha\right]$

So, we get $\mathbb{E}[|U|^\alpha]^2 \ge 1$ which implies $\mathbb{E}[|U|^\alpha] \ge 1$ since $|U|^\alpha > 0$

4. Let $N = 1_{\{X_1 > X_2\}} + 1_{\{X_n > X_{n-1}\}} + \sum_{i=2}^{n-1} 1_{\{X_i > X_{i-1}, X_i > X_{i+1}\}}$.

a) $$\mathbb{E}[N] = P\{X_1 > X_2\} + P\{X_n > X_{n-1}\} + \sum_{i=2}^{n-1} P\{X_i > X_{i-1}, X_i > X_{i+1}\}$$

$$= \frac{1}{2} + \frac{1}{2} + \sum_{i=2}^{n-1} \frac{1}{3}$$

$$= 1 + \frac{n-2}{3}$$

$$= \frac{n+1}{3}$$

$$\mathrm{Var}(N) = \mathrm{Var}(1_{\{X_1>X_2\}}) + \mathrm{Var}(1_{\{X_n>X_{n-1}\}}) + \sum_{i=2}^{n-1} \mathrm{Var}(1_{\{X_i>X_{i-1},\, X_i>X_{i+1}\}})$$

$$+2\,\mathrm{Cov}(1_{\{X_1>X_2\}}, 1_{\{X_n>X_{n-1}\}}) + 2\sum_{i=2}^{n-1} \mathrm{Cov}(1_{\{X_1>X_2\}}, 1_{\{X_i>X_{i-1},\, X_i>X_{i+1}\}})$$

$$+2\sum_{i=2}^{n-1} \mathrm{Cov}(1_{\{X_n>X_{n-1}\}}, 1_{\{X_i>X_{i-1},\, X_i>X_{i+1}\}})$$

$\rightarrow$ $\mathrm{Var}(1_{\{X_1>X_2\}}) = P\{X_1>X_2\} - P\{X_1>X_2\}^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\rightarrow$ $\mathrm{Var}(1_{\{X_n>X_{n-1}\}}) = \frac{1}{4}$

$\rightarrow$ $\mathrm{Var}(1_{\{X_i>X_{i-1},\, X_i>X_{i+1}\}}) = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$

$\rightarrow$ $\mathrm{Cov}(1_{\{X_1>X_2\}}, 1_{\{X_n>X_{n-1}\}}) = P\{X_1>X_2,\, X_n>X_{n-1}\} - P\{X_1>X_2\}P\{X_n>X_{n-1}\} = \frac{1}{4} - \frac{1}{4} = 0$

$\rightarrow$ <u>If $i=2$:</u> $\mathrm{Cov}(1_{\{X_1>X_2\}}, 1_{\{X_2>X_1,\, X_2>X_3\}}) = 0 - \frac{1}{2}\cdot\frac{1}{3} = -\frac{1}{6}$

$\rightarrow$ <u>If $i=3$:</u> $\mathrm{Cov}(1_{\{X_1>X_2\}}, 1_{\{X_3>X_2,\, X_3>X_4\}}) = P\{X_1>X_2,\, X_3>X_2,\, X_3>X_4\} - P\{X_1>X_2\} * P\{X_3>X_2,\, X_3>X_4\}$

$*$ $P\{X_1>X_2,\, X_3>X_2,\, X_3>X_4\} = P\{X_1>X_2,\, X_3>X_2>X_4\} + P\{X_1>X_2,\, X_3>X_4>X_2\}$

$$= P\{X_1>X_3>X_2>X_4\} + P\{X_3>X_1>X_2>X_4\} + P\{X_1>X_3>X_4>X_2\}$$

$$+P\{X_3>X_1>X_4>X_2\} + P\{X_3>X_4>X_1>X_2\}$$

$= \frac{5}{4!} = \frac{5}{24}$ since all orderings are equally likely.

So, $\mathrm{Cov}(1_{\{X_1>X_2\}}, 1_{\{X_3>X_2,\, X_3>X_4\}}) = \frac{5}{24} - \frac{1}{2}\cdot\frac{2}{9} = \frac{7}{72}$

$\rightarrow$ <u>If $i\geq 4$:</u> $\mathrm{Cov}(1_{\{X_1>X_2\}}, 1_{\{X_i>X_{i-1},\, X_i>X_{i+1}\}}) = P\{X_1>X_2,\, X_i>X_{i-1},\, X_i>X_{i+1}\} - P\{X_1>X_2\} * P\{X_i>X_{i-1},\, X_i>X_{i+1}\}$

$*$ $P\{X_1>X_2,\, X_i>X_{i-1},\, X_i>X_{i+1}\} = P\{X_1>X_2,\, X_i>X_{i-1}>X_{i+1}\} + P\{X_1>X_2,\, X_i>X_{i+1}>X_{i-1}\}$

$$= P\{X_1>X_2 \mid X_i>X_{i-1}>X_{i+1}\}P\{X_i>X_{i-1}>X_{i+1}\}$$

$$+P\{X_1>X_2 \mid X_i>X_{i+1}>X_{i-1}\}P\{X_i>X_{i+1}>X_{i-1}\}$$

$$= \frac{1}{2}\cdot\frac{2}{9} + \frac{1}{2}\cdot\frac{2}{9} = \frac{2}{9}$$

So, Cov(1

$\rightarrow \text{Cov}\left(1_{\{X_n > X_{n-1}\}}, 1_{\{X_i > X_{i-1}, X_i > X_{i+1}\}}\right) = \frac{1}{9}$, by symmetry

Thus,

$$Var(N) = \frac{1}{4} + \frac{1}{4} + (n-2)\frac{2}{9} + 2\left(-\frac{1}{6} + \frac{7}{72} + (n-4)\cdot\frac{1}{9}\right) + 2\left(-\frac{1}{6} + \frac{7}{72} + (n-4)\cdot\frac{1}{9}\right)$$

$$= \frac{1}{2} + \frac{2(n-2)}{9} + 4\left(-\frac{5}{72} + \frac{n-4}{9}\right)$$

$$= \frac{1}{2} + \frac{2(n-2)}{9} - \frac{5}{18} + \frac{4(n-4)}{9}$$

$$= \frac{2}{9} + \frac{2n-4+4n-16}{9}$$

$$= \frac{6n-18}{9}$$

$$= \frac{2n}{3} - 2$$

b) $E[N] = P\{X_1 > X_2\} + P\{X_n > X_{n-1}\} + \sum_{i=2}^{n-1} P\{X_i > X_{i-1}, X_i > X_{i+1}\}$

$$= \frac{\binom{9}{2}}{9^2} + \frac{\binom{9}{2}}{9^2} + \sum_{i=2}^{n-1}\left[P\{X_i > X_{i-1} > X_{i+1}\} + P\{X_i > X_{i+1} > X_{i-1}\}\right]$$

$$= \frac{9(9-1)}{2 \cdot 9^2} * 2 + \sum_{i=2}^{n-1} 2P\{X_i > X_{i-1} > X_{i+1}\}$$

$$= \frac{9-1}{9} + 2(n-2)\cdot\frac{\binom{9}{3}}{9^3}$$

$$= \frac{9-1}{9} + 2(n-2)\frac{9(9-1)(9-2)}{6 \cdot 9^3}$$

$$= \frac{9-1}{9} + (n-2)\frac{(9-1)(9-2)}{3 \cdot 9^2}$$

$$Var(N) = Var\left(1_{\{X_1 > X_2\}}\right) + Var\left(1_{\{X_n > X_{n-1}\}}\right) + \sum_{i=2}^{n-1} Var\left(1_{\{X_i > X_{i-1}, X_i > X_{i+1}\}}\right)$$

$$+ 2\,\text{Cov}\left(1_{\{X_1 > X_2\}}, 1_{\{X_n > X_{n-1}\}}\right) + 2\sum_{i=2}^{n-1} \text{Cov}\left(1_{\{X_1 > X_2\}}, 1_{\{X_i > X_{i-1}, X_i > X_{i+1}\}}\right)$$

$$+ 2\sum_{i=2}^{n-1} \text{Cov}\left(1_{\{X_n > X_{n-1}\}}, 1_{\{X_i > X_{i-1}, X_i > X_{i+1}\}}\right).$$

$\rightarrow Var\left(1_{\{X_1 > X_2\}}\right) = P\{X_1 > X_2\} - P\{X_1 > X_2\}^2 = \frac{9-1}{29}\left(1 - \frac{9-1}{29}\right) = \frac{(9-1)(9+1)}{49^2}$

$\rightarrow Var\left(1_{\{X_n > X_{n-1}\}}\right) = \frac{(9-1)(9+1)}{\ldots}$ by symmetry.

$\rightarrow \operatorname{Var}(1_{\{X_i > X_{i-1}, X_i > X_{i+1}\}}) = P\{X_i > X_{i-1}, X_i > X_{i+1}\} - P\{X_i > X_{i+1}, X_i > X_{i+1}\}^2$

$$= \frac{(q-1)(q-2)}{3q^2}\left(1 - \frac{(q-1)(q-2)}{3q^2}\right)$$

$$= \frac{(q-1)(q-2)}{3q^2} \quad \frac{(2q-1)(q+2)}{3q^2}$$

$$= \frac{(2q-1)(q-1)(q-2)(q+2)}{9q^4}$$

$\rightarrow \operatorname{Cov}(1_{\{X_1 > X_2\}}, 1_{\{X_n > X_{n-1}\}}) = 0$ since $X_1, \ldots, X_n$ are chosen independently.

$\rightarrow$ If $i=2$: $\operatorname{Cov}(1_{\{X_1 > X_2\}}, 1_{\{X_2 > X_1, X_2 > X_3\}}) = 0 - \frac{q-1}{2q} \quad \frac{(q-1)(q-2)}{3q^2}$

$$= -\frac{(q-1)^2(q-2)}{6q^3}$$

$\rightarrow$ If $i=3$ $\operatorname{Cov}(1_{\{X_1 > X_2\}}, 1_{\{X_3 > X_2, X_3 > X_4\}}) = P\{X_1 > X_2, X_3 > X_2, X_3 > X_4\} - P\{X_1 > X_2\} *$
$P(X_3 > X_2, X_3 > X_4\}$

$* P(X_1 > X_2, X_3 > X_2, X_3 > X_4) = P(X_1 > X_2, X_3 > X_2 > X_4) + P(X_1 > X_2, X_3 > X_4 > X_2)$
$+ P(X_1 > X_2, X_3 > X_2 = X_4)$

b) We know from (a), $\mathbb{E}[N] = \frac{n+1}{3}$. For any $\varepsilon > 0$,

$$P\left(\left|\frac{N}{n} - \frac{1}{3}\right| > \varepsilon\right) = P\left(\left|N - \frac{n}{3}\right| > n\varepsilon\right)$$

Chebychev's Inequality

$$\leq \frac{\mathbb{E}\left[\left(N - \frac{n}{3}\right)^2\right]}{n^2\varepsilon^2}$$

$$= \frac{\mathbb{E}\left[\left(N - \frac{n+1}{3} + \frac{1}{3}\right)^2\right]}{n^2\varepsilon^2}$$

$$= \frac{\mathbb{E}\left[\left(N - \frac{n+1}{3}\right)^2\right] + \frac{2}{3}\mathbb{E}\left[N - \frac{n+1}{3}\right] + \frac{1}{9}}{n^2\varepsilon^2}$$

$$= \frac{Var(N) + \frac{1}{9}}{n^2\varepsilon^2}$$

$$= \frac{2n}{3n^2\varepsilon^2} - \frac{2}{n^2\varepsilon^2} + \frac{1}{9n^2\varepsilon^2}$$

$$= \frac{2}{3n\varepsilon} - \frac{2}{n^2\varepsilon^2} + \frac{1}{9n^2\varepsilon^2}$$

Taking limit as $n \to \infty$, $P\left(\left|\frac{N}{n} - \frac{1}{3}\right| > \varepsilon\right) \to 0$ as $n \to \infty$. Thus $\frac{N}{n} \to \frac{1}{3}$ in probability as $n \to \infty$.