Spring 2007, Applied Probability

1. Same as #3 in Fall 2013

2. Let us find probability generating function of Z.

$$G_Z(s) = E[s^Z] = \sum_{k=0}^{\infty} s^k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!} = e^{-\lambda} \cdot e^{s\lambda} = e^{\lambda(s-1)}.$$

Then let us find moment generating function of Z

$$M_Z(t) = E[e^{tZ}] = \sum_{k=0}^{\infty} e^{tk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

a) $E[Z^2] = M_Z''(0)$

$$M_Z'(t) = e^{\lambda(e^t-1)} \lambda e^t = \lambda e^{\lambda(e^t-1)+t}$$

$$M_Z''(t) = \lambda e^{\lambda(e^t-1)+t}(\lambda e^t+1) = \lambda^2 e^{\lambda(e^t-1)+2t} + \lambda e^{\lambda(e^t-1)+t}$$

So, $E[Z^2] = \lambda^2 + \lambda$

b) $E[Z^3] = M_Z'''(0)$

$$M_Z'''(t) = \lambda^2 e^{\lambda(e^t-1)+2t}(\lambda e^t+2) + \lambda e^{\lambda(e^t-1)+t}(\lambda e^t+1)$$

$$E[Z^3] = \lambda^2(\lambda+2) + \lambda(\lambda+1) = \lambda^3 + 2\lambda^2 + \lambda^2 + \lambda = \lambda^3 + 3\lambda^2 + \lambda$$

c) $$E[Y] = E[Z!] = \sum_{k=0}^{\infty} k! e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=0}^{\infty} e^{-\lambda} \lambda^k = e^{-\lambda} \sum_{k=0}^{\infty} \lambda^k, \ e^{-\lambda} \cdot \frac{1}{1-\lambda} = \frac{e^{-\lambda}}{1-\lambda},$$

since $0<\lambda<1$.

d) $E[X] = E[2^Z] = G_Z(2) = e^{\lambda(2-1)} = e^{\lambda}$

e) $Var(X) = E[X^2] - E[X]^2$

$\rightarrow E[X^2] = E[(2^Z)^2] = E[4^Z] = G_Z(4) = e^{\lambda(4-1)} = e^{3\lambda}$

So, $Var(X) = e^{3\lambda} - e^{2\lambda}$.

3. We have $\mu = E[X] = 10$, $\sigma^2 = Var(X) = 7$ and $A = \{X>0\}$

a) For any r.v. X,

$$P(|X - E[X]| \geq \varepsilon) \leq \frac{Var(X)}{\varepsilon^2}$$

b) By Chebychev's Inequality, we have for any $\varepsilon > 0$.

$$P(|X-10| \geq \varepsilon) \leq \frac{7}{\varepsilon^2} \Rightarrow P(|X-10| < \varepsilon) \geq 1 - \frac{7}{\varepsilon^2}$$

$$\Rightarrow P(X > 10-\varepsilon) \geq P(|X-10| < \varepsilon) \geq 1 - \frac{7}{\varepsilon^2}$$

Set $\varepsilon = 10$ to get

$$P(X>0) \leq 1 - \frac{7}{10^2} = 0.93$$

c) For the r.v. X and Y,

$$\mathbb{E}[XY]^2 \leq \mathbb{E}[X^2]\,\mathbb{E}[Y^2]$$

d) We have $\mathbb{E}[X 1_{\{X>0\}}]^2 \leq \mathbb{E}[X^2]\,\mathbb{E}[1^2_{\{X>0\}}] = \mathbb{E}[X^2]\,P(A)$

$\rightarrow$ $\mathbb{E}[X 1_{\{X>0\}}] = \mathbb{E}[X] - \underbrace{\mathbb{E}[X 1_{\{X=0\}}]}_{=0 \text{ since } X 1_{\{X=0\}} = 0}$ since $X \geq 0$ by its definition.

So, $\mathbb{E}[X 1_{\{X>0\}}] = 10$

$\rightarrow$ $\mathbb{E}[X^2] = Var(X) + (\mathbb{E}[X])^2 = 7 + 100 = 107$

Substitution into the inequality yields

$$107\,P(A) \geq 100 \quad \Rightarrow \quad P(A) \geq \frac{100}{107}$$