Spring 2005, Applied Probability:

1. The joint density of $(X_{(i)}, X_{(j)})$ for $i<j$ is as follows

$$f_{X_{(i)},X_{(j)}}(u,v) = \frac{n!}{(i-1)!\,(n-j-1)!\,(n-j)!}[F(u)]^{i-1} f(u)\,[F(v)-F(u)]^{j-i-1} f(v)\,[1-F(v)]^{n-j}$$

2. Firstly consider that (assuming X_i's and N are all independent)

$$\mathbb{E}[Yg(Y)\,|\,N=n] = \mathbb{E}\Big[\sum_{i=1}^{n} X_i\, g\Big(\sum_{i=1}^{n} X_i\Big)\,\Big|\,N=n\Big] = \sum_{i=1}^{n} \mathbb{E}\Big[X_i\, g\Big(\sum_{j=1}^{n} X_j\Big)\Big]$$

Then,

$$\mathbb{E}[Yg(Y)] = \mathbb{E}[\mathbb{E}(Yg(Y)|N)]$$

$$= \sum_{k=1}^{\infty} e^{-\lambda}\cdot\frac{\lambda^k}{k!}\Big(\sum_{i=1}^{k} \underbrace{\mathbb{E}\Big[X_i\, g\Big(\sum_{j=1}^{k} X_j\Big)\Big]}_{=\mathbb{E}\left[X_0\, g\left(\sum_{j=1}^{k-1} X_j + X_0\right)\right]}\Big)$$

since X_i's are identically distributed

$$\big(= \mathbb{E}[X_0\, g(X_0)] \text{ if } k=1\big)$$

$$= \sum_{k=1}^{\infty} e^{-\lambda}\,\frac{\lambda^k}{k!}\, k\, \mathbb{E}\Big[X_0\, g\Big(\sum_{j=1}^{k-1} X_j + X_0\Big)\Big]$$

$$= \lambda \sum_{k=1}^{\infty} e^{-\lambda}\cdot\frac{\lambda^{k-1}}{(k-1)!}\, \mathbb{E}\Big[X_0\, g\Big(\sum_{j=1}^{k-1} X_j + X_0\Big)\Big]$$

$$= \lambda \sum_{k=0}^{\infty} e^{-\lambda}\cdot\frac{\lambda^{k}}{k!}\, \mathbb{E}\Big[X_0\, g\Big(\sum_{j=1}^{k} X_j + X_0\Big)\Big]$$

$$= \lambda\, \mathbb{E}\big[\mathbb{E}[X_0\, g(Y+X_0)]\,|\,N\big]$$

$$= \lambda\, \mathbb{E}[X_0\, g(Y+X_0)]$$

3. Let us denote the distance between the midpoint of the stick and the point where it is broken by Y. Also, let ℓ be the length of the stick. Then,

$$X = \frac{\ell/2 - Y}{\ell/2 + Y} = -1 + \frac{\ell}{\ell/2 + Y}$$

Since the stick is broken uniformly, we see that $Y \sim \text{Uniform}[0, \ell/2]$.

Now, consider

$$\mathbb{E}[X] = -1 + \ell\, \mathbb{E}\left[\frac{1}{\ell/2 + Y}\right]$$

$$= -1 + \ell \int_0^{\ell/2} \frac{1}{\ell/2 + y} \cdot \frac{2}{\ell}\, dy$$

$$= -1 + 2\left[\log(\ell/2 + y)\right]_0^{\ell/2}$$

$$= -1 + 2\left(\log(\ell) - \log(\ell/2)\right)$$

$$= -1 + 2\log 2$$

$$\mathbb{E}[X^2] = 1 + 2\ell\, \mathbb{E}\left[\frac{1}{\ell/2 + Y}\right] + \ell^2\, \mathbb{E}\left[\frac{1}{(\ell/2 + Y)^2}\right]$$

$$= -1 + 4\log 2 + \ell^2 \int_0^{\ell/2} \frac{1}{(\ell/2 + y)^2} \frac{2}{\ell}\, dy$$

$$= -1 + 4\log 2 + 2\ell\left[-\frac{1}{\ell/2 + y}\right]_0^{\ell/2}$$

$$= -1 + 4\log 2 + 2$$

$$= 1 + 4\log 2$$

So, $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \cancel{1} + 4\log 2 - \cancel{1} + 4\log 2 + 4(\log 2)^2 = 4\log 2\,(2 + \log 2)$

4. Let N be the number of electrons that hit the plate where $N \sim \text{Poisson}(2)$ and let T_i be the number of secondary electrons produced by ith electron where $T_i \sim \text{Poisson}(1)$. Let T be the total number of secondary electrons. So, we have $T = \sum_{i=1}^{N} T_i$.

a) $\rightarrow \mathbb{E}[e^{tT_i}] = \sum_{k=0}^{\infty} e^{tk} e^{-1} \frac{1}{k!} = e^{-1} \sum_{k=0}^{\infty} \frac{(e^t)^k}{k!} = e^{e^t - 1}$

$\rightarrow \mathbb{E}[s^N] = \sum_{k=0}^{\infty} s^k e^{-2} \frac{2^k}{k!} = e^{-2} \sum_{k=0}^{\infty} \frac{(2s)^k}{k!} = e^{2s-2}$

$\rightarrow \mathbb{E}[e^{tT} | N=n] = \mathbb{E}\left[e^{t\sum_{i=1}^{n} T_i} \,\middle|\, N=n\right] \overset{\text{Independence}}{=} \prod_{i=1}^{n} \mathbb{E}[e^{tT_i}] = (e^{e^t - 1})^n$

So, $\mathbb{E}[e^{tT} | N] = (e^{e^t - 1})^N$.

$\rightarrow \mathbb{E}[e^{tT}] = \mathbb{E}\left[\mathbb{E}[e^{tT} | N]\right] = \mathbb{E}\left[(e^{e^t - 1})^N\right] = e^{2(e^{e^t - 1}) - 2}$

Thus, $M_T(t) = e^{2(e^{e^t-1})-2}$

b) Remember that $M_X^{(n)}(0) = \mathbb{E}[X^n]$

$\rightarrow M_T'(t) = e^{2(e^{e^t-1})-2} \cdot 2e^{e^t-1} e^t$

$= 2e^{2e^{e^t-1}+e^t+t-3}$

$\rightarrow \mathbb{E}[T] = M_T'(0) = 2$

$\rightarrow M_T''(t) = 2e^{2e^{e^t-1}+e^t+t-3}\left(2e^{e^t-1}e^t + e^t+1\right)$

$\rightarrow \mathbb{E}[T^2] = M_T''(0) = 2e^{2+1-3}(2+1+1) = 8$

$\rightarrow \mathrm{Var}(T) = \mathbb{E}[T^2] - \mathbb{E}[T]^2 = 8-4 = 4$

OR: Since T_i's are iid Poisson(1), we see that $T \sim$ Poisson(N). So,

$\rightarrow \mathbb{E}[T] = \mathbb{E}\left[\mathbb{E}[T \mid N]\right] = \mathbb{E}[N] = 2$

$\rightarrow \mathbb{E}[T^2] = \mathbb{E}\left[\mathbb{E}[T^2 \mid N]\right] = \mathbb{E}[N + N^2] = 2+2+4 = 8$

$\rightarrow \mathrm{Var}(T) = \mathbb{E}[T^2] - \mathbb{E}[T]^2 = 8-4 = 4$