Qualifying Exam 505a. February, 2003

1. Let X and Y be independent binomial with parameters N, p and M, p, respectively.

(i) Find the distribution of $X+Y$,

(ii) Find the conditional distribution of X given $X+Y$.

2. In a town of $N+1$ inhabitants a person tells a rumor to a second person, who in turn repeats it to a third person , and so on. At each step the recipient of the rumor is chosen at random from the N inhabitants available.

(i) Find the probability that the rumor will be told (transferred to a person) n times without, (a) returning to the originator, (b) being repeated to any person;

(ii) The rumor mongers constitute $100p\%$ of the population of a large town, which is to say that n, the number of times the rumor was told, is equal to $p(N+1)$ where $N+1$ is the total population of the town. In problem (i), find the limit of the probability in question as $N \longrightarrow \infty$.

3. (i) Let X_1 and X_2 be random variables with joint density function

$$f(x_1, x_2) = \begin{cases} \frac{1}{4}\left[1 + x_1 x_2 (x_1^2 - x_2^2)\right] & \text{if } |x_1| \leq 1, |x_2| \leq 1, \\ & 0 \text{ otherwise.} \end{cases}$$

and $Z = X_1 + X_2$. Prove that the characteristic functions of X_1, X_2, and Z verify $\phi_Z(t) = \phi_{X_1}(t)\,\phi_{X_2}(t)$.

(ii). Are X_1 and X_2 independent? Prove.

HINT: Take advantage of symmetries to avoid excessive calculation.

4. (i) Show that for $t > 0$ and $x \in \mathbf{R}$, for every r.v. X, $P(X \geq x) \leq e^{-tx} E e^{tX}$.

(ii) Let $X_1, X_2, \ldots$be i.i.d r.v.'s with the distribution function $F_{X_1}(x) = 1 - e^{-x}, x \geq 0$; and $S_n = X_1 + \ldots + X_n$.Show that

$$P\left(\frac{S_n}{n} > 1 + \varepsilon\right) \leq e^{-[\varepsilon - \log(1+\varepsilon)]n}.$$

Hint: Use (i) to solve (ii).