MATH 505A QUALIFYING EXAM
FEBRUARY 12, 2002

You should try at least 3 problems; you may try all 4.

(1) Let $X_1, X_2, ..$ be iid with characteristic function $\varphi(t) = e^{-|t|^\alpha}$, where $0 < \alpha < 2$.
(a) Show that $\frac{X_1+..+X_n}{n^{1/\alpha}}$ has the same distribution as X_1.
(b) Show that $\text{var}(X_1) = \infty$. HINT: Use (a).
(c) Suppose $\alpha < 1$. Show that the weak law of large numbers does not hold, that is, there is no constant μ such that $\frac{X_1+..+X_n}{n} \to \mu$ in probability.

(2) The Geophysics building at the University of Northern California is scheduled to be seismically reinforced. The reinforcement will occur at a random time uniformly distributed in the next 3 years. Suppose that during any fixed time interval of length t, the number of major earthquakes is Poisson with mean λt. Find the probability that no major earthquake occurs before the reinforcement of the Geophysics building.

(3) A worm farm operates as follows. Let Z_n be the number of worms at the end of month n, with $Z_0 = 1$. Each month, each worm present at the start of the month dies and is replaced by a Binomial(2,p) number of offspring; these numbers of offspring are independent from one worm to another. In addition, with probability r the worm farmer buys one new worm during the month and adds it to the farm, independently of what his current worms are doing. (In other words, he adds a Bernoulli(r) number of worms.)
(a) Let $G_n(s)$ be the generating function of Z_n. Find an equation relating G_{n+1} to G_n.
(b) Suppose $p = r$. Find an explicit formula for $G_2(s)$.

(4) 8 people, including 4 members of the Smith family and 4 members of the Jones family, divide themselves at random (meaning all outcomes are equally likely) into 4 pairs of partners, to play chess. Let N be the number of Smiths whose partners are also Smiths. Find the mean and variance of N. HINT: One approach is to use indicators.