505a Qualifying exam. November 17, 1999

Do problems 1,2, and **one more problem, chosen from 3,4,5, or 6.**

1.) Let $X, X_1, X_2, \cdots$ be iid with $\mathbb{P}(X > t) = e^{-t}$ for $t > 0$. Let $S_n = X_1 + \cdots + X_n$.

a) Simplify $\mathbb{E}e^{\beta X}$ for $\beta \in (-\infty, 1)$. [You may calculate formally, without rigorous justification.]

b) Simplify $\mathbb{E}X^k$, for $k = 1, 2, 3, \ldots$.

[STRATEGY for c) and d) together: these are the special cases $n = 1, 2$ of a general fact about the order statistics of n independent uniforms constructed from $n+1$ independent exponentials, and you may do the general case once, in place of c) and d). But if d) is too hard, be sure to do c) anyway.]

c) Show that $X_1/(X_1 + X_2)$ is uniformly distributed on [0,1].

d) Let U, V be independent uniform on [0,1]. Let $A = \min(U, V)$ and $B = \max(U, V)$. Show that

$$(A, B) \text{ and } \left(\frac{X_1}{S_3}, \frac{X_1 + X_2}{S_3}\right)$$

have the same joint distribution.

2.) Let $p \in (0, 1)$, let $X_1, X_2, \ldots$ be iid with $\mathbb{P}(X_i = 1) = p, \mathbb{P}(X_i = 0) = 1 - p$, and let $S_n = X_1 + \cdots + X_n$.

a) Simplify a1) $\mathbb{E}S_n$, a2) $\text{VAR}(S_n)$, a3) $\mathbb{E}z^{S_n}$.

b) Let $\lambda > 0$, let N be Poisson with parameter λ, with N and S_n independent for every n. Let $Y = S_N, Z = N - Y$ so that $Y + Z = N$. Show that Y and Z are independent.

c) Let $\lambda, p_1, \ldots, p_r > 0$ with $p_1 + \cdots + p_r = 1$. Let Z_i be Poisson with $\mathbb{E}Z_i = \lambda p_i$ and $Z_1, Z_2, \ldots, Z_r$ independent. Let $N = Z_1 + \cdots + Z_r$. Show that the distribution of the vector $\mathbf{Z} := (Z_1, \ldots, Z_r)$, conditional on the event $N = n$, is multinomial with parameters $(n; p_1, \ldots, p_r)$, for each $n = 0, 1, 2, \ldots$.

d.) For N Poisson with parameter λ and fixed $a > 1$ calculate

$$\lim \frac{\mathbb{P}(N \geq k)}{\mathbb{P}(N = k)},$$

with the limit taken for $\lambda \to \infty, k \to \infty, k/\lambda \to a$. [HINT: geometric series.]

3.) Let π be a random permutation of $\{1, 2, \ldots, n\}$, with all $n!$ possibilities equally likely. Let W be the number of fixed points of π, i.e. $W := \sum_1^n X_i$, where X_i is the indicator of the event that $\pi_i = i$.

a) Compute $\mathbb{E}W$ and $\mathbb{E}W^2$ and simplify, exactly.

b) Show that $\mathbb{P}(W=0) \to e^{-1}$ as $n \to \infty$.

c) Extend b) by showing that for $k = 0, 1, 2, ...$, $\mathbb{P}(W = j) \to e^{-1}1^j/j!$ as $n \to \infty$. [HINT: you may use the version of inclusion-exclusion known as Waring's formula, which has the form $\mathbb{P}(W=j) = \sum_{k\geq 0}(-1)^k\binom{j+k}{k}s_{j+k}$.]

4.) n people each roll one fair die. For each (unordered) pair of people that get the same number of spots, that number of spots is scored, with S for the total score achieved among the $\binom{n}{2}$ pairs of people. For example, if there are $n = 10$ people, and they roll 1,2,2,2,3,4,4,4,4,6 then $S = 2+2+2+4+4+4+4+4+4$ since there are three pairs of people matching 2 and six $= \binom{4}{2}$ pairs of people scoring 4. [HINT: Consider S as the sum of $\binom{n}{2}$ random variables $S_{i,j}$, where $S_{i,j}$ is k if persons i and j both roll k, and zero otherwise.]

a) Simplify $\mathbb{E}S$.

b) Simplify $\mathbb{E}S^2$.

5.) Let $S_0, S_1, S_2, \dots$ be simple symmetric random walk, i.e. $\mathbb{P}(S_i - S_{i-1} = 1) = \mathbb{P}(S_i - S_{i-1} = -1) = 1/2$, with independent increments. Let $T = \min n > 0 : S_n = 0$ be the hitting time to zero. Write P_a for probabilities for the walk starting with $S_0 = a$.

a) What does the reflection principle say about $\mathbb{P}_a(S_n = i, T \leq n)$, for $a > 0$, and $i, n \geq 0$?

b) What does the reflection principle say about $\mathbb{P}_a(S_n \geq i, T > n)$, for $a > 0$, and $i, n \geq 0$? [Hint: telescoping series]

c) For fixed $a > 0$, give asymptotics for $\mathbb{P}_a(T > n)$ as $n \to \infty$. [HINT: Stirling's formula is that $n! \sim \sqrt{2\pi n}(n/e)^2$.]

d) Simplify, for fixed $a > 0$, $\lim_{n\to\infty} \mathbb{P}_{a+1}(T > n)/\mathbb{P}_a(T > n)$.

6. Let $X, X_1, X_2, \dots$ be iid standard normal. Let M_n be the maximum of $X_1, \dots, X_n$. Let Y have the extreme value distribution, with $\mathbb{P}(L < c) = e^{-e^{-c}}$ for $c \in (-\infty, \infty)$. You will have established a distributional limit of the form $(M_n - a_n)/b_n \Rightarrow L$ by showing that, for $c \in (-\infty, \infty)$

$$\mathbb{P}\left(M_n < \sqrt{2\log n}\left(1 + \frac{c - \log(\sqrt{2\log n}\ \sqrt{2\pi})}{2\log n}\right)\right) \to e^{-e^{-c}}$$

You may use, without proof, the fact that $1 - \Phi(x) \sim \phi(x)/x$ as $x \to \infty$, where $\Phi(x) := \mathbb{P}(X \leq x)$ and $\phi = \Phi'$ is the standard normal density. [HINT: recall that $(1 - a_n)^n \to e^{-t}$ if $na_n \to t$; use this twice.]