Math 505a qualifying exam (Fall 2025)

Answer all 4 questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all four problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

Problem 1:

a) If X takes non-negative integer values, show that

$$\mathbb{E}\,(X) = \sum_{n\geq 1}^{\infty} P(X \geq n).$$

b) Let X and Y be independent random variables taking values in the non-negative integers, with finite means. Let $U = min(X, Y)$ and $V = max(X, Y)$. Show (part a may be helpful) that

$$\mathbb{E}\,(U) = \sum_{r\geq 1} P(X \geq r)P(Y \geq r)$$

$$\mathbb{E}\,(V) = \sum_{r\geq 1} [P(X \geq r) + P(Y \geq r) - P(X \geq r)P(Y \geq r)]$$

$$\mathbb{E}\,(UV) = \sum_{r,s\geq 1} P(X \geq r)P(Y \geq s)$$

Problem 2:

(i) Alice and Bob plan to meet in a two-hour session between 8:00am and 10:00am. Each arrives independently at a random time and then waits for 30 minutes. What is the probability that they meet?

(ii) Let X and Y be two independent random variables with uniform distribution on the unit interval $[0, 1]$ such that they separate the unit interval into three pieces. What is the probability that the three pieces can be constructed into a triangle?

Problem 3:

Consider a simple random walk $S_n = X_1 + \cdots + X_n$ starting at $S_0 = a > 0$, where $a \in \mathbb{N}$ and X_i are i.i.d. Rademacher random variables, i.e., $P(X_i = \pm 1) = 1/2$. What is the probability that it hits $b \in \mathbb{N}$ before hitting 0 when $b > a$? **Do NOT solve this problem with the martingale optional stopping argument.**

Problem 4:

Your initial capital is 1 unit, and every time "dear leader" speaks, this gets halved, i.e., multiplied by .5, with probability p, or else tripled, with probability $1-p$. These effects are iid, and independent of the number of times she speaks. Suppose X, the number of times she speaks, is Poisson distributed, with mean 100. Let Y be the number of times your capital got halved, and Z be the number of times it got tripled. Note that $X = Y + Z$.

a) What is the distribution of Y?

b) Are Y and Z independent of each other?

c) $B := .5^Y$ is the bad impact on your capital, due to halvings. Simplify $\mathbb{E}(B)$.

d) $G := 3^Z$ is the good impact on your capital, due to triplings. Simplify $\mathbb{E}(G)$.

e) What must p be so that BG, the combined effect of good and bad, has mean 1 — so that your expected capital, after dear leader is done, is unchanged?