Qualifying Exam: Applied Probability

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1. Let X be a random variable taking values in $\mathbb{Z}^+$ with probability mass function

$$\mathbb{P}(X = n) = \frac{1}{(e-1)n!}.$$

a. Find $\mathbb{E}(u^X)$.

Solution. By the law of the unconscious statistician,

$$\mathbb{E}(u^X) = \sum_{n=1}^{\infty} u^n \, \mathbb{P}(X = n) = \frac{1}{e-1} \sum_{n=1}^{\infty} \frac{u^n}{n!} = \frac{e^u - 1}{e-1}.$$

b. Find $\mathbb{E}(X)$ and $\text{var}(X)$.

Solution. By direct computation,

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} n \, \mathbb{P}(X = n) = \frac{1}{e-1} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \frac{1}{e-1} \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{e}{e-1}.$$

To find the variance of X, we compute the factorial moment

$$\mathbb{E}(X(X-1)) = \frac{1}{e-1} \sum_{n=1}^{\infty} \frac{n(n-1)}{n!} = \frac{1}{e-1} \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = \frac{e}{e-1},$$

which allows us to compute

$$\text{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \mathbb{E}(X(X-1)) + \mathbb{E}(X) - \mathbb{E}(X)^2 = \frac{e(e-2)}{(e-1)^2}.$$

c. Let $(U_i)_{i \geq 1}$ be a sequence of i.i.d. Uniform($[0,1]$) random variables independent of X. Find the cumulative distribution function of $M := \max\{U_1, \ldots, U_X\}$.

Solution. Since M takes values in $[0,1]$, consider $m \in [0,1]$. We compute

$$\begin{aligned}
\mathbb{P}(M \leq m \mid X = n) &= \mathbb{P}(U_1 \leq m, \ldots, U_X \leq m \mid X = n) \\
&= \mathbb{P}(U_1 \leq m, \ldots, U_n \leq m \mid X = n) \\
&= \mathbb{P}(U_1 \leq m, \ldots, U_n \leq m) \\
&= \mathbb{P}(U_1 \leq m) \cdots \mathbb{P}(U_n \leq m) \\
&= m^n.
\end{aligned}$$

By the law of total probability and by part (a), $\mathbb{P}(M \leq m) = \sum_{n=1}^{\infty} m^n \, \mathbb{P}(X = n) = \mathbb{E}(m^X) = (e^m - 1)/(e-1)$. We conclude that

$$\mathbb{P}(M \leq m) = \begin{cases} 0 & \text{if } m < 0, \\ \frac{e^m - 1}{e-1} & \text{if } 0 \leq m \leq 1, \\ 1 & \text{if } m > 1. \end{cases}$$

2. a. Let $Z_1, Z_2, \dots$ be i.i.d. Exponential(λ) random variables, and let $S_n = Z_1 + \cdots + Z_n$. Find the joint probability density function of $(S_1/S_{n+1}, \dots, S_n/S_{n+1})$.

Hint: Show that the joint probability density function of $(S_1, \dots, S_{n+1})$ depends only on s_{n+1}. Use this to find the conditional distribution of $(S_1, \dots, S_n)$ given $S_{n+1} = s_{n+1}$.

Solution. We follow the hint. Since $(s_1, s_2, \dots, s_{n+1}) \mapsto (s_1, s_2 - s_1, \dots, s_{n+1} - s_n)$ is a linear transformation (skew transformation) with Jacobian determinant 1, the following holds for $s_1 < \cdots < s_{n+1}$:

$$\begin{aligned} f_{S_1,S_2,\dots,S_{n+1}}(s_1, s_2, \dots, s_{n+1}) &= f_{Z_1,Z_2,\dots,Z_{n+1}}(s_1, s_2 - s_1, \dots, s_{n+1} - s_n) \\ &= f_{Z_1}(s_1) \cdot f_{Z_2}(s_2 - s_1) \cdots f_{Z_{n+1}}(s_{n+1} - s_n) \\ &= \lambda e^{-\lambda s_1} \cdot \lambda e^{-\lambda(s_2 - s_1)} \cdots \lambda e^{-\lambda(s_{n+1} - s_n)} \\ &= \lambda^{n+1} e^{-\lambda s_{n+1}}. \end{aligned}$$

Induction yields $f_{S_{n+1}}(s_{n+1}) = \lambda^{n+1} s_{n+1}^n e^{-\lambda s_{n+1}}/n!$, and hence

$$\begin{aligned} f_{S_1,\dots,S_n|S_{n+1}}(s_1, \dots, s_n \mid s_{n+1}) &= \frac{f_{S_1,\dots,S_{n+1}}(s_1, \dots, s_{n+1})}{f_{S_{n+1}}(s_{n+1})} \\ &= \frac{n!}{s_{n+1}^n} \cdot \mathbb{1}\{0 < s_1 < \cdots < s_n < s_{n+1}\}. \end{aligned}$$

It follows that

$$\begin{aligned} f_{S_1/S_{n+1},\dots,S_n/S_{n+1}|S_{n+1}}(t_1, \dots, t_n \mid s_{n+1}) &= s_{n+1}^n \cdot f_{S_1,\dots,S_n|S_{n+1}}(t_1 s_{n+1}, \dots, t_n s_{n+1} \mid s_{n+1}) \\ &= n! \cdot \mathbb{1}\{0 < t_1 < \cdots < t_n < 1\}, \\ f_{S_1/S_{n+1},\dots,S_n/S_{n+1}}(t_1, \dots, t_n) &= n! \cdot \mathbb{1}\{0 < t_1 < \cdots < t_n < 1\}. \end{aligned}$$

Remark. In other words, $(S_1/S_{n+1}, \dots, S_n/S_{n+1})$ has the same distribution as the order statistics of n i.i.d. Uniform$(0,1)$ random variables.

b. Let $U_1, \dots, U_5$ be i.i.d. Uniform$(0,1)$ random variables. Find $\mathbb{P}(U_{(1)} + U_{(5)} \le 2U_{(3)})$.

Hint: Use part (a).

Solution. Because $f_{U_{(1)},\dots,U_{(5)}}(t_1, \dots, t_5) = 5! \cdot \mathbb{1}\{0 < t_1 < \cdots < t_5 < 1\}$, part (a) implies

$$(U_{(1)}, \dots, U_{(5)}) \stackrel{\text{d}}{=} \left(\frac{S_1}{S_6}, \dots, \frac{S_5}{S_6}\right).$$

This allows us to compute

$$\begin{aligned} \mathbb{P}(U_{(1)} + U_{(5)} \le 2U_{(3)}) &= \mathbb{P}(S_1 + S_5 \le 2S_3) \\ &= \mathbb{P}(2Z_1 + Z_2 + Z_3 + Z_4 + Z_5 \le 2Z_1 + 2Z_2 + 2Z_3) \\ &= \mathbb{P}(Z_4 + Z_5 \le Z_2 + Z_3), \end{aligned}$$

which is $\frac{1}{2}$ by symmetry ($Z_4 + Z_5$ and $Z_2 + Z_3$ are i.i.d. continuous random variables).

3. There are $n \geq 6$ people, numbered from 1 to n; each wears a hat with their number written on it. The people throw their hats into a pile, and then each takes a hat from the pile, with all $n!$ permutations equally likely.

   Let N be the number of pairs $\{i, j\}$ such that person i has hat j, and person j has hat i. Let M be the number of triplets $\{k, \ell, m\}$ such that person k has hat ℓ, person ℓ has hat m, and person m has hat k.

   a. Find $\mathbb{E}(N)$ and $\mathbb{E}(M)$.

   Solution. Call the given random permutation F, and observe that

$$N = \sum_{1 \leq i < j \leq n} \mathbb{1}\{F(i) = j, F(j) = i\}.$$

   By the linearity of expectation,

$$\mathbb{E}(N) = \sum_{1 \leq i < j \leq n} \mathbb{P}(F(i) = j, F(j) = i) = \sum_{1 \leq i < j \leq n} \frac{(n-2)!}{n!} = \binom{n}{2} \cdot \frac{(n-2)!}{n!} = \frac{1}{2}.$$

   Similarly,

$$M = \sum_{1 \leq k < \ell < m \leq n} \mathbb{1}\{F(k) = \ell, F(\ell) = m, F(m) = k\},$$

$$\mathbb{E}(M) = \sum_{1 \leq k < \ell < m \leq n} \mathbb{P}(F(k) = \ell, F(\ell) = m, F(m) = k) = \binom{n}{3} \cdot \frac{(n-3)!}{n!} = \frac{1}{6}.$$

   b. Find $\operatorname{cov}(N, M)$.

   Solution. By the distributive property,

$$NM = \sum_{i<j} \sum_{k<\ell<m} \mathbb{1}\{F(i) = j, F(j) = i, F(k) = \ell, F(\ell) = m, F(m) = k\}.$$

   Call the above indicator $I_{i,j,k,\ell,m}$. Because permutations are *bijections* from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$, we can have $I_{i,j,k,\ell,m} = 1$ only if i, j, k, ℓ, m are distinct. There are $\binom{n}{5}\binom{5}{2}$ ways to choose such distinct i, j, k, ℓ, m, so, by the linearity of expectation,

$$\mathbb{E}(NM) = \sum_{i<j} \sum_{k<\ell<m} \mathbb{E}(I_{i,j,k,\ell,m}) = \binom{n}{5}\binom{5}{2} \cdot \frac{(n-5)!}{n!} = \frac{1}{12}.$$

   By part (a), $\operatorname{cov}(N, M) = \mathbb{E}(NM) - \mathbb{E}(N)\,\mathbb{E}(M) = 0$.

   c. Are N and M independent?

   Solution. No. Observe that $\mathbb{P}(N = \frac{n}{2}) > 0$ and $\mathbb{P}(M = \frac{n}{3}) > 0$, but $\mathbb{P}(N = \frac{n}{2}, M = \frac{n}{3}) = 0$.