Qualifying Exam: Applied Probability

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Answer all three questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem, you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

1. a. Let X be a nonnegative random variable with finite expected value. Show that

$$\sum_{i=1}^{\infty} \mathbb{P}(X \geq i) \leq \mathbb{E}(X) \leq 1 + \sum_{i=1}^{\infty} \mathbb{P}(X \geq i).$$

Solution. Because $\mathbb{P}(X \geq x)$ is a decreasing function of x,

$$\mathbb{P}(X \geq i) \geq \int_{i}^{i+1} \mathbb{P}(X \geq x)\,\mathrm{d}x \geq \mathbb{P}(X \geq i+1).$$

Taking the sum over all $i \in \mathbb{N}$ yields

$$1 + \sum_{i=1}^{\infty} \mathbb{P}(X \geq i) \geq \mathbb{E}(X) \geq \sum_{i=1}^{\infty} \mathbb{P}(X \geq i),$$

where $\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X \geq x)\,\mathrm{d}x$ is the tail-sum formula for a nonnegative random variable, a consequence of Tonelli's theorem.

Remark. We cannot assume that X is continuous, i.e., that X admits a probability density function.

b. Show that if X takes values in $\{0, \ldots, n\}$ for some nonnegative integer n, then

$$\sum_{i=1}^{\infty} \mathbb{P}(X \geq i) = \mathbb{E}(X).$$

Solution. Exchanging the order of summation of finitely many terms yields

$$\sum_{i=1}^{\infty} \mathbb{P}(X \geq i) = \sum_{i=1}^{\infty}\sum_{j=i}^{\infty} \mathbb{P}(X = j) = \sum_{i=1}^{n}\sum_{j=i}^{n} \mathbb{P}(X = j) = \sum_{j=1}^{n}\sum_{i=1}^{j} \mathbb{P}(X = j) = \sum_{j=1}^{n} j\,\mathbb{P}(X = j) = \mathbb{E}(X).$$

c. Let M be the minimum value seen in 4 rolls of a fair die. Find $\mathbb{E}(M)$.

Solution. By part (b),

$$\mathbb{E}(M) = \sum_{m=1}^{6} \mathbb{P}(M \geq m) = \left(\frac{6}{6}\right)^4 + \cdots + \left(\frac{1}{6}\right)^4 = \frac{2275}{1296}.$$

2. Let X and Y be independent random variables with distribution $\text{Uniform}([0,1])$.

   a. Find the probability density function of $X+2Y$.

   Solution. By direct computation,

$$\begin{aligned} f_{X+2Y}(z) &= \int_0^1 f_X(x)\cdot f_{2Y}(z-x)\,\mathrm{d}x \\ &= \int_0^1 f_X(x)\cdot \frac{1}{2} f_Y\left(\frac{z-x}{2}\right)\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 \mathbb{1}\{z-x\in[0,2]\}\,\mathrm{d}x \\ &= \begin{cases} z/2 & \text{if } 0\le z\le 1, \\ 1/2 & \text{if } 1\le z\le 2, \\ (3-z)/2 & \text{if } 2\le z\le 3. \end{cases} \end{aligned}$$

   b. Find the joint probability density function of $X-Y$ and $X+Y$.

   Solution. Write $U=X+Y$ and $V=X-Y$, and observe that $X=(U+V)/2$ and $Y=(U-V)/2$. Then, by direct computation,

$$\begin{aligned} f_{U,V}(u,v) &= \left|\det\begin{pmatrix} \frac{\partial}{\partial u}\frac{u+v}{2} & \frac{\partial}{\partial v}\frac{u+v}{2} \\ \frac{\partial}{\partial u}\frac{u-v}{2} & \frac{\partial}{\partial v}\frac{u-v}{2} \end{pmatrix}\right|\cdot f_{X,Y}\left(\frac{u+v}{2},\frac{u-v}{2}\right) \\ &= \frac{1}{2}\cdot f_{X,Y}\left(\frac{u+v}{2},\frac{u-v}{2}\right) \\ &= \begin{cases} 1/2 & \text{if } u+v, u-v\in[0,2] \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

3. Let $0 < p < 1$, and consider a sequence of independent trials, each with the same success probability of p. For each $n \geq 1$, let p_n be the probability that there are an odd number of successes in the first n trials.

   a. Express p_n in terms of p_{n-1}.

   Solution. By conditioning on the outcome of the nth trial, we see that

$$p_n = p \cdot (1 - p_{n-1}) + (1 - p) \cdot p_{n-1}.$$

   b. Based on part (a), find the value of λ such that if $p_{n-1} = \lambda$, then $p_n = \lambda$.

   Solution. If $p_{n-1} = \lambda$, then, by part (a),

$$\begin{aligned} p_n &= p \cdot (1 - \lambda) + (1 - p) \cdot \lambda \\ &= p + \lambda - 2p\lambda, \end{aligned}$$

   which is equal to λ if and only if $\lambda = 1/2$.

   c. Using the value of λ found in part (b), show that $\lim_{n\to\infty} p_n = \lambda$.

   Hint: Write $p_n = \lambda + \varepsilon_n$.

   Solution. Following the hint, we can rewrite the equation we found in part (a) as

$$\begin{aligned} \lambda + \varepsilon_n &= p \cdot (1 - (\lambda + \varepsilon_{n-1})) + (1 - p) \cdot (\lambda + \varepsilon_{n-1}) \\ &= \lambda + (1 - 2p) \cdot \varepsilon_{n-1}, \end{aligned}$$

   which yields $\varepsilon_n = (1 - 2p)^{n-1} \cdot \varepsilon_1$. Because $|1 - 2p| < 1$, we conclude that

$$\lim_{n\to\infty} p_n = \lambda + \lim_{n\to\infty} \varepsilon_n = \lambda.$$