Fall 2014, Applied Probability :

1. Suppose A and B are independent.

We can write $\xi = 5\,1_A - 7\,1_{A^c}$ and $\eta = 2\,1_B + 3\,1_{B^c}$. So, we have $\xi\eta = 10\,1_{A\cap B} + 15\,1_{A\cap B^c} - 14\,1_{A^c\cap B} - 21\,1_{A^c\cap B^c}$. Then,

$$\mathbb{E}[\xi\eta] = 10P(A\cap B) + 15P(A\cap B^c) - 14P(A^c\cap B) - 21P(A^c\cap B^c)$$

On the other hand,

$$\mathbb{E}[\xi] * \mathbb{E}[\eta] = (5P(A) - 7P(A^c))(2P(B) + 3P(B^c))$$

$$= 10P(A\cap B) + 15P(A\cap B^c) - 14P(A^c\cap B) - 21P(A^c\cap B^c) \quad \text{by using independence}$$

So, we see that $\mathbb{E}[\xi\eta] = \mathbb{E}[\xi]\mathbb{E}[\eta] < \infty$ which means that, ξ and η are uncorrelated.

Conversely, suppose that ξ and η are uncorrelated. Then define

$$\alpha(\omega) = \xi(\omega) + 7 = \begin{cases} 12 & \text{if } \omega\in A \\ 0 & \text{if } \omega\notin A \end{cases}, \quad \beta(\omega) = \eta(\omega) - 3 = \begin{cases} -1 & \text{if } \omega\in B \\ 0 & \text{if } \omega\notin B \end{cases}$$

Then, we have

$$(\alpha\beta)(\omega) = \begin{cases} -12 & \text{if } \omega\in A\cap B \\ 0 & \text{otherwise} \end{cases}$$

Now, observe that

$$\mathbb{E}[\xi\eta] = \mathbb{E}[(\alpha-7)(\beta+3)] = \mathbb{E}[\alpha\beta] + 3\mathbb{E}[\alpha] - 7\mathbb{E}[\beta] - 21$$

$$\mathbb{E}[\xi]\mathbb{E}[\eta] = (\mathbb{E}[\alpha]-7)(\mathbb{E}[\beta]+3) = \mathbb{E}[\alpha]\mathbb{E}[\beta] + 3\mathbb{E}[\alpha] - 7\mathbb{E}[\beta] - 21$$

Since $\mathbb{E}[\xi\eta] = \mathbb{E}[\xi]\mathbb{E}[\eta]$ $(<\infty)$ we obtain that $\mathbb{E}[\alpha\beta] = \mathbb{E}[\alpha]\mathbb{E}[\beta]$.

We have

$$\mathbb{E}[\alpha\beta] = -12P(A\cap B) \quad \text{and} \quad \mathbb{E}[\alpha]\mathbb{E}[\beta] = 12P(A)(-1)P(B) = -12P(A)P(B)$$

So, we get $P(A\cap B) = P(A)P(B)$ which shows that A and B are independent.

2. Let $S = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} S_{i,j}$ where $S_{i,j}$ is k if persons i and j both roll k, and 0 otherwise.

a) Firstly consider

$$\mathbb{E}[S_{i,j}] = \sum_{m=1}^{6} m\frac{1}{36} = \frac{7}{12}$$

Then, $\mathbb{E}[S] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbb{E}[S_{ij}] = \frac{7}{12} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = \frac{7}{12} \cdot \frac{n(n-1)}{2} = \frac{7n(n-1)}{24}$

b) Let $R_i := \sum_{j=i+1}^{n} S_{i,j}$ Then $S = \sum_{i=1}^{n-1} R_i$ and

$$\mathbb{E}[S^2] = \sum_{i=1}^{n-1} \mathbb{E}[R_i^2] + 2 \sum_{1 \le i < j \le n-1} \mathbb{E}[R_i R_j]$$

$$\longrightarrow \mathbb{E}[R_i^2] = \sum_{j=i+1}^{n} \mathbb{E}[S_{i,j}^2] + 2 \sum_{i+1 \le j < k \le n} \mathbb{E}[S_{i,j} S_{i,k}]$$

$$* \ \mathbb{E}[S_{i,j}^2] = \sum_{m=1}^{6} m^2 \frac{1}{36} = \frac{91}{36}$$

$$* \ \mathbb{E}[S_{i,j} S_{i,k}] = \sum_{m=1}^{6} m^2 \frac{1}{216} = \frac{91}{216}$$

So, $\mathbb{E}[R_i^2] = (n-i) \frac{91}{36} + (n-i)(n-i-1) \cdot \frac{91}{216} = (n-i) \frac{91}{36} \left(1 + (n-i-1) \cdot \frac{1}{6}\right)$

$\longrightarrow$ $\mathbb{E}[R_i R_j]$ is too complicated!

3. a) $$M_{Y_a}(t) = \int_{-\infty}^{\infty} e^{xt} f_{Y_a}(x)\,dx = \int_{-\infty}^{-a} e^{xt} f_{-X}(x)\,dx + \int_{-a}^{a} e^{xt} f_X(x)\,dx + \int_{a}^{\infty} e^{xt} f_{-X}(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{xt} f_X(x)\,dx \quad , \text{since } -X \sim N(0,1)$$

$$= M_X(t) \quad \text{where } X \sim N(0,1).$$

Thus Y_a is a standard normal r.v.

b) Consider

$$XY_a = \begin{cases} X^2, & \text{if } |X| < a \\ -X^2, & \text{if } |X| > a \end{cases}$$

$$\mathbb{E}[XY_a] = \underbrace{\int_{-\infty}^{-a} (-x^2)\varphi(x)\,dx}_{= \int_a^{\infty} -x^2 \varphi(x)\,dx} + \underbrace{\int_{-a}^{a} x^2 \varphi(x)\,dx}_{1 - 2\int_0^{\infty} x^2 \varphi(x)\,dx} + \int_{a}^{\infty} (-x^2)\varphi(x)\,dx = 1 - 4\int_{a}^{\infty} x^2 \varphi(x)\,dx$$

c) Consider that $\lim_{a \to \infty} \varphi(a) = 1$ and $\lim_{a \to -\infty} \varphi(a) = -3$. By continuity of $\varphi(a)$, we conclude that $\exists a \in \mathbb{R}$ s.t. $\varphi(a) = 0$.

c) Consider the region $(0,a)\times(-a,0) \in \mathbb{R}^2$, and

$$P((X,Y_a)\in(0,a)\times(-a,0)) = P(X\in(0,a),\ Y_a\in(-a,0))$$

$$= P(Y_a\in(-a,0)\mid X\in(0,a))\,P(X\in(0,a))$$

$$= P(X\in(-a,0)\mid X\in(0,a))\,P(X\in(0,a))$$

$$= 0$$

So, $P((X,Y_a)\in(0,a)\times(-a,0)) = 0$ which would be impossible if (X,Y_a) had bivariate normal distribution.