Fall 2013, Applied Probability

1. a) Firstly, consider the probability of none of A and B occurs in a trial

$$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - a - b.$$

Now, let A_k be the event that A occurs before B for the first time at k^{th} trial. So, $P(A_k) = (1-a-b)^{k-1} a$ and $A = \bigcup_{k=1}^{\infty} A_k$ where A_k's are mutually exclusive. So,

$$P(A) = \sum_{k=1}^{\infty} P(A_k) = \sum_{k=1}^{\infty} (1-a-b)^{k-1} a = a \cdot \frac{1}{a+b} = \frac{a}{a+b}$$

b) Let A: the event that sum is 3 and B: the event that sum is 7. So,

$P(A) = \frac{1}{18}$ and $P(B) = \frac{1}{6}$. So, the probability that sum 3 will occur before sum 7 is

$$\frac{1/18}{1/18 + 1/6} = \frac{1}{4}$$

2. a) We see that W and Z are linear combinations of independent standard normal r.v. X and Y. So, they are bivariate normal. In this case, to show independence, it is enough to show W and Z are uncorrelated.

$$\left.\begin{aligned} &E[WZ] = E[X^2] - E[Y^2] = 0 \\ &E[W]\,E[Z] = 0 \end{aligned}\right\} \quad Cov(W,Z) = E[WZ] - E[W]E[Z] = 0$$

Thus, W and Z are independent.

b) Consider that $X + 2Y = X + Y + Y = W + Y$ and since $Y = \frac{W-Z}{2}$, we have

$X + 2Y = W + \frac{W-Z}{2} = \frac{3}{2}W - \frac{1}{2}Z$. So,

$$E[X+2Y \mid Z] = \frac{3}{2} E[W \mid Z] - \frac{1}{2} E[Z \mid Z] = \frac{3}{2} E[W] - \frac{1}{2} Z$$ (by independence and property of conditional expectation)

$$= -\frac{1}{2} Z$$

c) Let A denote the event "$X>0$". Then,

$$E[X \mid X>0] = \frac{E[X 1_A]}{P(A)} = 2E[X 1_A] = 2\int_0^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$u = -\frac{x^2}{2}$

$du = -x\,dx$

$$= -2\int_0^{-\infty} \frac{1}{\sqrt{2\pi}} e^u du = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{0} e^u du = \sqrt{\frac{2}{\pi}}$$

3 Let $X = \sum_{i=1}^{d} 1_{A_i}$ where A_i is the event that ith box is empty.

a) $$\mathbb{E}[X] = \sum_{i=1}^{d} P(A_i) = \sum_{i=1}^{d} \left(\frac{d-1}{d}\right)^n = \frac{(d-1)^n}{d^{n-1}}$$

b) $$\operatorname{Var} X = \sum_{i=1}^{d} \operatorname{Var}(1_{A_i}) + 2 \sum_{1 \le i < j \le d} \operatorname{Cov}(1_{A_i}, 1_{A_j})$$

$$\to \operatorname{Cov}(1_{A_i}, 1_{A_j}) = \mathbb{E}[1_{A_i} 1_{A_j}] - \mathbb{E}[1_{A_i}]\,\mathbb{E}[1_{A_j}]$$
$$= P(A_i \cap A_j) - P(A_i)P(A_j)$$
$$= \left(\frac{d-2}{d}\right)^n - \left(\frac{d-1}{d}\right)^{2n}$$

So, $$\operatorname{Var} X = \sum_{i=1}^{d} \left[\left(\frac{d-1}{d}\right)^n - \left(\frac{d-1}{d}\right)^{2n}\right] + 2 \sum_{1 \le i < j \le d} \left[\left(\frac{d-2}{d}\right)^n - \left(\frac{d-1}{d}\right)^{2n}\right]$$
$$= \frac{(d-1)^n}{d^{n-1}} - \frac{(d-1)^{2n}}{d^{2n-1}} + \frac{(d-1)(d-2)^n}{d^{n-1}} - \frac{(d-1)^{2n}}{d^{2n-1}}$$

c) $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$P(A) = P(B) = P(C) = \left(\frac{d-4}{d}\right)^n, \quad P(A \cap B) = P(B \cap C) = \left(\frac{d-6}{d}\right)^n, \quad P(A \cap C) = \left(\frac{d-8}{d}\right)^n$$

$$P(A \cap B \cap C) = \left(\frac{d-8}{d}\right)^n$$

So, $$P(A \cup B \cup C) = 3\left(\frac{d-4}{d}\right)^n - 2\left(\frac{d-6}{d}\right)^n = \frac{3(d-4)^n - 2(d-6)^n}{d^n}$$

d) $$P(D) = \frac{1 \cdot (d-1)(d-2)\cdots(d-n+1)}{d^n} = \frac{d(d-1)(d-2)\cdots(d-n+1)(d-n)!}{d^{n+1}(d-n)!} = \frac{d!}{d^{n+1}(d-n)!}$$

Using Stirling's formula,

$$P(D) \approx \sqrt{2\pi d}\left(\frac{d}{e}\right)^d \frac{1}{d^{n+1}} \frac{1}{\sqrt{2\pi(d-n)}} \frac{1}{\left(\frac{d-n}{e}\right)^{d-n}}$$
$$= \sqrt{\frac{d}{d-n}} \left(\frac{d}{e}\right)^{d-n} \left(\frac{d}{e}\right)^n \frac{1}{d^{n+1}} \left(\frac{e}{d-n}\right)^{d-n}$$
$$= \sqrt{\frac{d}{d-n}} \frac{1}{d} e^{-n} \left(\frac{d}{e} \cdot \frac{e}{d-n}\right)^{d-n}$$
$$= \frac{e^{-n}}{\sqrt{d(d-n)}} \left(\frac{d}{d-n}\right)^{d-n}$$