Math 505a 2012 Fall Qualifying Exam

1. In the Polya Urn model, $w \geq 1$ white balls and $b \geq 1$ black balls are placed in an urn at time 0, and at times $1, 2, \ldots$ a ball is chosen uniformly from the urn independent of the past, and replaced back into the urn with one additional ball of the same color.

a. A vector $(X_1, \ldots, X_n)$ of random variables is said to be exchangeable

$$(X_1, \ldots, X_n) =_d (X_{\pi(1)}, \ldots, X_{\pi(n)}) \quad \text{for all permutations } \pi$$

where $=_d$ denotes equality in distribution. If X_i is the indicator that a white ball is drawn from the urn at time i, prove that $(X_1, \ldots, X_n)$ is exchangeable.

b. Find the mean and variance of $S_n = X_1 + \cdots + X_n$, the total number of white balls added to the urn up to time n.

2. With a and b positive numbers, a needle of length $l \in (0, \min(a, b)]$ is dropped randomly on a rectangular grid consisting of an infinite number of parallel lines distance a apart, and, perpendicular to these, an infinite number of parallel lines distance b apart. Let A and B, respectively, be the events that the needle intersects the group of lines at distance a and b apart.

a. Show $P(A) = \frac{2l}{a\pi}$ and $P(B) = \frac{2l}{b\pi}$. Hint: The angle θ giving the orientation of the needle might be taken as uniform from $[0, 2\pi)$, but by symmetry, one may assume that the angle is uniformly taken from $[0, \pi/2)$.

b. Determine $P(A \cap B)$ and verify that A and B are strictly negatively correlated, that is, that $P(A \cap B) < P(A)P(B)$.

3. A total of k boys and $n - k$ girls sit around a circular table, with all $\cancel{n!}$ $(n-1)!$ arrangements equally likely. Compute the mean and variance of the number Y of pairs of boy/girl neighbors. Note: In the circular arrangement GBGGBB, since the first G and last B are neighbors, $Y = 4$.

Fall 2012, Applied Probability

1. a) Consider probability $P(X_1 = x_1, \ldots, X_n = x_n)$ where $x_1, \ldots, x_n \in \{0,1\}$.
Then, this probability is the probability of having exactly $\sum_{i=1}^{n} x_i$ white balls among chosen n balls. So, the order of white balls does not affect the probability Thus for any π, $P(X_{\pi(1)} = x_1, \ldots, X_{\pi(n)} = x_n)$ represents the probability of getting $\sum_{i=1}^{n} x_i$ white balls among chosen n balls. Thus,

$$(X_1, \ldots, X_n) \stackrel{d}{=} (X_{\pi(1)}, \ldots, X_{\pi(n)})$$

for all permutations π.

b) Let $S_n = X_1 + \cdots + X_n$ represent the total number of balls added to the urn up to time n. Here,

$$X_i = \begin{cases} 1 & \text{if a white ball added at time } i \\ 0 & \text{otherwise} \end{cases}$$

$$\rightarrow \mathbb{E} S_n = \sum_{i=1}^{n} \mathbb{E} X_i = \sum_{i=1}^{n} P\{X_i = 1\}$$

Now, consider

- $P(X_1 = 1) = \dfrac{w}{w+b}$

- $P(X_2 = 1) = P(X_2 = 1 \mid X_1 = 1) P(X_1 = 1) + P(X_2 = 1 \mid X_1 = 0) P(X_1 = 0) \qquad (*)$

$$= \frac{w+1}{w+b+1} \cdot \frac{w}{w+b} + \frac{w}{w+b+1} \cdot \frac{b}{w+b}$$

$$= \frac{w(w+b+1)}{(w+b)(w+b+1)}$$

$$= \frac{w}{w+b}$$

- Observe above that we can replace 2 by $i+1$ and 1 by i in $(*)$ above, since $P(X_1 = 1) = P(X_2 = 1)$ (inductively). Then, we obtain

$$P(X_i = 1) = \frac{w}{w+b}$$

Thus, we obtain

$$\mathbb{E}[S_n] = \frac{nw}{w+b}$$

$$\rightarrow \mathrm{Var}(S_n) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{1 \le i < j \le n} \mathrm{Cov}(X_i, X_j)$$

- $\mathrm{Var}(X_i) = P\{X_i = 1\} - P\{X_i = 1\}^2 = \frac{w}{w+b}\left(1 - \frac{w}{w+b}\right) = \frac{wb}{(w+b)^2}$

- $\mathrm{Cov}(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j] = P\{X_i = 1, X_j = 1\} - P\{X_i = 1\}P\{X_j = 1\}$

  - For $i < j$,

$$\left.\begin{aligned} P\{X_i = 1, X_j = 1\} &= P\{X_j = 1 \mid X_i = 1\} P\{X_i = 1\} \\ &= P\{X_2 = 1 \mid X_1 = 1\} P\{X_1 = 1\} \end{aligned}\right\} \text{by the same reasoning in part (a).}$$

$$= \frac{w+1}{w+b+1} \quad \frac{w}{w+b}$$

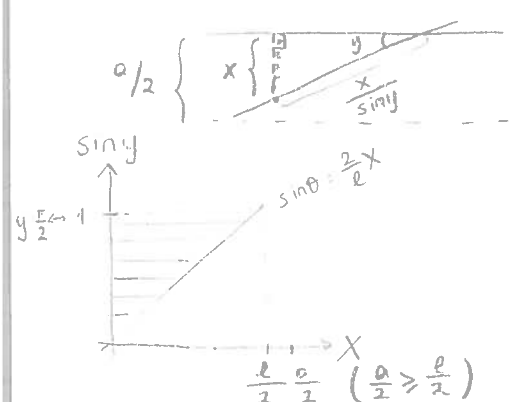
$$\mathrm{Cov}(X_i, X_j) = \frac{(w+1)w}{(w+b+1)(w+b)} - \frac{w^2}{(w+b)^2} = \frac{w}{w+b}\left(\frac{w+1}{w+b+1} - \frac{w}{w+b}\right)$$

$$\mathrm{Var}(S_n) = \frac{nwb}{(w+b)^2} + \frac{n(n-1)w}{w+b}\left(\frac{w+1}{w+b+1} - \frac{w}{w+b}\right).$$

2 a) θ : the accute angle between the needle and the horizontal lines (wlog)
X : the distance between the midpoint of the needle and horizontal lines

Assume also wlog that the distance between horizontal lines is a and the distance between vertical lines is b

$\theta \sim U[0, \frac{\pi}{2})$ and $X \sim U[0, \frac{a}{2})$ by symmetry



$$\Rightarrow P(A) = P\left(\frac{X}{\sin y} < \frac{\ell}{2}\right)$$

$$= \int_0^{\pi/2} \int_0^{\ell \sin y/2} f_X(x) f_\theta(y)\, dx\, dy = \frac{4}{a\pi} \int_0^{\pi/2} \frac{\ell}{2} \sin y\, dy$$

$$= \frac{2\ell}{a\pi} \left[-\cos y\right]_0^{\pi/2} = \frac{2\ell}{a\pi}$$

By changing horizontal's to vertical's, a's to b's and keeping θ the same, the above argument works for $P(B)$ in exactly the same way. Thus, we get $P(B) = \frac{2\ell}{b\pi}$.

b) Now, consider

$$P(A \cap B) = P\left(\frac{X}{\cos\theta} < \frac{\ell}{2}, \frac{Y}{\sin\theta} < \frac{\ell}{2}\right)$$

$$= P\left(X < \frac{\ell\cos\theta}{2}, Y < \frac{\ell\sin\theta}{2}\right)$$

$$= \int_0^{\pi/2}\int_0^{\frac{\ell\sin\theta}{2}}\int_0^{\frac{\ell\cos\theta}{2}} \frac{2}{a}\,\frac{2}{b}\,\frac{2}{\pi}\,dx\,dy\,d\theta$$

$$= \frac{8}{ab\pi}\int_0^{\pi/2} \frac{\ell^2\sin 2\theta}{8}\,d\theta$$

$$= \frac{\ell^2}{ab\pi}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\ell^2}{ab\pi}\left[\frac{1}{2}+\frac{1}{2}\right]$$

$$= \frac{\ell^2}{ab\pi}$$

In this case, $P(A \cap B) < P(A)P(B) \iff \frac{\ell^2}{ab\pi} < \frac{4\ell^2}{ab\pi^2} \iff 1 < \frac{4}{\pi}$ which is always true So, we have $P(A \cap B) < P(A)P(B)$