Fall 2011, Applied Probability:

1 Consider by definition,

$$P(B|A) = P(B|A^c) \Rightarrow \frac{P(A\cap B)}{P(A)} = \frac{P(A^c \cap B)}{P(A^c)}$$

$$\Rightarrow \frac{P(A\cap B)}{P(A)} = \frac{P(B) - P(A\cap B)}{1 - P(A)}$$

$$\Rightarrow P(A\cap B) - \cancel{P(A)P(A\cap B)} = P(A)P(B) - \cancel{P(A)P(A\cap B)}$$

$$\Rightarrow P(A\cap B) = P(A)P(B)$$

Thus, the events A and B are independent.

2. a) $F_Z(z) = P(Z \leq z) = P(X - Y \leq z)$

→ If $z \geq 0$:

$$F_Z(z) = \int_0^\infty \int_0^{y+z} e^{-x} e^{-y}\, dx\, dy = \int_0^\infty e^{-y} \left[-e^{-x}\right]_0^{y+z} dy = \int_0^\infty e^{-y}\left(-e^{-y-z}+1\right) dy$$

$$= \int_0^\infty \left(e^{-y} - e^{-2y-z}\right) dy = \left[-e^{-y} + \frac{e^{-z} e^{-2y}}{2}\right]_0^\infty = 1 - \frac{e^{-z}}{2}$$

→ If $z < 0$:

$$F_Z(z) = \int_{-z}^\infty \int_0^{y+z} e^{-x} e^{-y}\, dx\, dy = \int_{-z} e^{-y} \left[-e^{-x}\right]_0^{y+z} dy = \int e^{-y}\left(-e^{-y-z}+1\right) dy$$

$$= \int_{-z}^\infty \left(e^{-y} - e^{-2y-z}\right) dy = \left[-e^{-y} + \frac{e^{-z} \cdot e^{-2y}}{2}\right]_{-z}^\infty = e^{z} - \frac{e^{z}}{2} = \frac{e^{z}}{2}$$

So,

$$F_Z(z) = \begin{cases} \frac{e^z}{2} & \text{if } z < 0 \\ 1 - \frac{e^{-z}}{2} & \text{if } z \geq 0 \end{cases}$$

and thus,

$$f_Z(z) = \begin{cases} \frac{e^z}{2} & \text{if } z < 0 \\ \frac{e^{-z}}{2} & \text{if } z \geq 0 \end{cases}$$

b) → $M_X(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} e^{-x}\, dx = \int_0^\infty e^{x(t-1)}\, dx = \left.\frac{e^{x(t-1)}}{t-1}\right|_0^\infty$

*we must have $t-1<0$, i.e. $\underline{t<1}$

$$= \frac{1}{t-1}(0-1) = \frac{1}{1-t} \quad \text{where } t < 1.$$

$$\longrightarrow \varphi_X(t) = \mathbb{E}\left[e^{itX}\right] = \int_0^\infty e^{itx} e^{-x} dx = \int_0^\infty e^{x(it-1)} dx = \left[\frac{e^{x(it-1)}}{it-1}\right]_0^\infty$$

$$= \frac{1}{it-1}\left[e^{-x}\cos tx + i e^{-x}\sin tx\right]_0^\infty = \frac{1}{it-1}(-1) = \frac{1}{1-it}$$

$$\longrightarrow \varphi_Z(t) = \mathbb{E}\left[e^{itZ}\right] = \int_{-\infty}^0 e^{itz}\cdot\frac{e^z}{2}dz + \int_0^\infty e^{itz}\cdot\frac{e^{-z}}{2}dz = \frac{1}{2}\int_{-\infty}^0 e^{z(it+1)}dz + \frac{1}{2}\int_0^\infty e^{z(it-1)}dz$$

$$= \frac{1}{2}\left[\frac{e^{z(it+1)}}{it+1}\right]_{-\infty}^0 + \frac{1}{2}\left[\frac{e^{z(it-1)}}{it-1}\right]_0^\infty = \frac{1}{2(it+1)} - \frac{1}{2(it-1)} = \frac{\cancel{it}-1-\cancel{it}-1}{2(-t^2-1)}$$

$$= \frac{1}{1+t^2}$$

3. Let us define $W = \sum_{i=1}^m \sum_{j=1}^n 1_{A_{ij}}$ where A_{ij} is the event that the birthday of i^{th} man and j^{th} women are the same (after numbering them).

a) $\mathbb{E}[W] = \sum_{i=1}^m \sum_{j=1}^n P(A_{ij})$

$$P(A_{ij}) : \sum_{k=1}^{365}(\text{bday of } i^{th} \text{ man} = \text{bday of } j^{th} \text{ women} = k) = \sum_{k=1}^{365}\frac{1}{(365)^2} = \frac{1}{365}$$

So, $\mathbb{E}[W] = \sum_{i=1}^m \sum_{j=1}^n \frac{1}{365} = \frac{mn}{365}$

b) Define $R_i := \sum_{k=1}^n 1_{A_{ik}}$. So, $W = \sum_{i=1}^m R_i$ and

$$Var(W) = \sum_{i=1}^m Var(R_i) + 2\sum_{1\le i<j\le m} Cov(R_i, R_j)$$

$$\longrightarrow Var(R_i) = \sum_{k=1}^n Var(1_{A_{ik}}) + 2\sum_{1\le k<l\le n} Cov(1_{A_{ik}}, 1_{A_{il}}).$$

$$\longrightarrow Var(1_{A_{ik}}) = P(A_{ik}) - P(A_{ik})^2 = \frac{1}{365}\left(1-\frac{1}{365}\right) = \frac{364}{(365)^2}$$

$$\longrightarrow Cov(1_{A_{ik}}, 1_{A_{il}}) = P(A_{ik}\cap A_{il}) - P(A_{ik})P(A_{il}) = \frac{1}{(365)^2} - \frac{1}{(365)^2} = 0, \quad \forall k, l.$$

$$\Rightarrow Var(R_i) = \frac{364n}{(365)^2}$$

$$\longrightarrow Cov(R_i, R_j) = Cov\left(\sum_{k=1}^n 1_{A_{ik}}, \sum_{l=1}^n 1_{A_{jl}}\right) = \sum_{k=1}^n \sum_{l=1}^n Cov(1_{A_{ik}}, 1_{A_{jl}})$$

$$\rightarrow \text{If } k=l : \quad Cov(1_{A_{ik}}, 1_{A_{jl}}) = P(A_{ik}\cap A_{jl}) - P(A_{ik})P(A_{jl}) = \frac{1}{(365)^2} - \frac{1}{(365)^2} = 0$$

$\rightarrow$ If $k \neq \ell$: $Cov(1_{A_{ik}}, 1_{A_{j\ell}}) = P(A_{ik} \cap A_{j\ell}) - P(A_{ik})P(A_{j\ell}) = \frac{1}{(365)^3} - \frac{1}{(365)^2}$

$$= \frac{1}{(365)^2} \cdot \frac{-364}{365} = \frac{-364}{(365)^3}$$

$$\Rightarrow Cov(R_i, \ell_j) = \frac{-364n(n-1)}{(365)^3}$$

$$\Rightarrow Var(W) = \sum_{i=1}^{m} \frac{364n}{(365)^2} + 2 \sum_{1 \leq i < j \leq m} \frac{-364n(n-1)}{(365)^3}$$

$$= \frac{364mn}{(365)^2} - \frac{364mn(m-1)(n-1)}{(365)^3}$$

$$= \frac{364mn}{(365)^2}\left[1 - \frac{(m-1)(n-1)}{365}\right]$$

$$= \frac{364mn(365 - (m-1)(n-1))}{(365)^3}$$

c) There are mn pairs to consider and if a pair has the same birthday, which has probability $1/365$, we can think this event to be success. We know that pairs are not independent, but approximately, we can say that $W \sim Binomial(mn, 1/365)$. Since $m=10$, $n=20$, we have $W \sim Binomial(200, 1/365)$. Then, we have $mn = 200 \geq 100$ and $p = \frac{1}{365} \leq 0.01$. So, actually, we can use Poisson approximation to Binomial distribution. So, we can say that $W \sim Poisson\left(\frac{200}{365}\right)$ where $\frac{200}{365} \approx 0.55$.

d) We want to find an approximate value for $P(W>0)$ or a bound for this probability. Consider

$$P(W>0) = 1 - P(W=0) = 1 - e^{-0.55} \cdot \frac{(0.55)^0}{0!} = 1 - e^{-0.55} < 1 - e^{-0.693} = 1 - e^{-\ln 2} = \frac{1}{2}$$

Thus, we have $P(W>0) < \frac{1}{2}$ showing that probability for the professor to win is less than $1/2$. So, in this case the professor should expect to lose money.