Fall 2010, Applied Probability:

1. Let $Y_n =: \frac{1}{n}\sum_{k=1}^{n} X_k$ and $Z_n = \frac{1}{n}\sum_{k=1}^{n} X_k^2$. Since X_k's are iid, by Law of Large Numbers, $Y_n \xrightarrow{P} E[X_k] = 1$ as $n \to \infty$. Similarly, $Z_n \xrightarrow{P} E[X_k^2] = Var(X_k) + E[X_k]^2 = 2$ as $n \to \infty$. Then, since $g = 1/x$ is continuous a.e. we can say that $g(Z_n) = \frac{1}{\frac{1}{n}\sum_{k=1}^{n} X_k^2} \xrightarrow{P} \frac{1}{2}$ as $n \to \infty$

Finally, we obtain that

$$\frac{\sum_{k=1}^{n} X_k}{\sum_{k=1}^{n} X_k^2} = \left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) \cdot \frac{1}{\frac{1}{n}\sum_{k=1}^{n} X_k^2} \xrightarrow{P} 1\ \frac{1}{2} \quad \text{as } n \to \infty$$

2 We have $E[kX_k] = p^k$ and $\sum_{k=1}^{\infty} p^k = \frac{p}{1-p} < \infty$, and $Var(kX_k) = kp^k$ and $\sum_{k=1}^{\infty} kp^k = \frac{p}{(1-p)^2} < \infty$ Since the sequence $\}Y_k\{_{k \geq 1}$ is independent, by Kolmogorov's Two Series Theorem, we conclude that $\sum_{k=1}^{\infty} kX_k$ converges a.s.

Now, let us find the distribution of the r.v. Y. Let X be a Poisson(λ) r.v. and consider

$$E[s^X] = \sum_{k=0}^{\infty} s^k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{\lambda(s-1)}$$

So, $E[s^{X_k}] = e^{\frac{p^k}{k}(s-1)}$ and $E[s^{kX_k}] = E[(s^k)^{X_k}] = e^{\frac{p^k}{k}(s^k-1)}$.

Then, define $Y_n = \sum_{k=1}^{n} kX_k$ and consider

$$E[s^{Y_n}] = E\left[\prod_{k=1}^{n} s^{kX_k}\right] = \prod_{k=1}^{n} e^{\frac{p^k}{k}(s^k-1)} = e^{\sum_{k=1}^{n}\frac{(ps)^k}{k} - \sum_{k=1}^{n}\frac{p^k}{k}}$$

Now, by taking limit as $n \to \infty$ (since we are allowed to take limit inside of expectation here), we get

$$E[s^Y] = e^{\sum_{k=1}^{\infty}\frac{(ps)^k}{k} - \sum_{k=1}^{\infty}\frac{p^k}{k}} = e^{-\ln(1-ps) + \ln(1-p)}, \quad -1 \leq ps < 1 \text{ and } -1 \leq p < 1$$

$$= \frac{1-p}{1-sp}, \quad -1 \leq sp < 1$$

Since we know $P\{X=k\} = \frac{G_X^{(k)}(0)}{k!}$, denoting the probability generating function of Y by $G(s)$, observe that

$$G(s) = \frac{1-p}{1-sp}, \quad G'(s) = (1-p)\frac{p}{(1-sp)^2}, \quad G''(s) = (1-p)\frac{2p^2}{(1-sp)^3},$$

$$G'''(s) = (1-p)\frac{3.2p^3}{(1-sp)^3}, \quad \ldots, \quad G^{(k)}(s) = (1-p)\frac{k!\,p^k}{(1-sp)^k}, \quad \text{for } k=0,1,2,\ldots$$

So, we get

$$P\{Y=k\} = (1-p)p^k, \quad k=0,1,2,\ldots$$

which gives us the distribution of Y.

3. a) Define

$$A = \begin{cases} 1 & \text{if the coin is flipped once and lands on head} \\ 0 & \text{otherwise} \end{cases}$$

We want to find $f_{U|A}(u|1)$, where $U \sim \text{Uniform}[0,1]$.

$$f_{U|A}(u|1) = \frac{f_{U,A}(u,1)}{f_A(1)} = \frac{f_{A|U}(1|u)\, f_U(u)}{f_A(1)} = \frac{u \cdot 1}{f_A(1)}$$

$$\rightarrow f_A(1) = \int_0^1 f_{A|U}(1|u)\, f_U(u)\,du = \int_0^1 u \cdot 1\,du = \left.\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$$

So, we obtain that

$$f_{U|A}(u|1) = 2u, \quad 0 \leq u \leq 1,$$

as the desired distribution.

b) In a similar way in part (a), define

$$B = \begin{cases} 1 & \text{if the coin is flipped 2000 times and lands on head 1500 times} \\ 0 & \text{otherwise} \end{cases}$$

Now, we want to find $f_{U|B}(u,1)$, where $U \sim \text{Uniform}[0,1]$

$$f_{U|B}(u|1) = \frac{f_{B|U}(1|u)\, f_U(u)}{f_B(1)} = \frac{\binom{2000}{1500} u^{1500} (1-u)^{500} \cdot 1}{f_B(1)}$$

$$f_B(1) = \int_0^1 f_{B|U}(1|u)\, f_U(u)\, du = \int_0^1 \binom{2000}{1500} u^{1500} (1-u)^{500}\, du$$

Then,

$$f_{U|B}(u|1) = \frac{\binom{2000}{1500} u^{1500} (1-u)^{500}}{\int_0^1 \binom{2000}{1500} u^{1500} (1-u)^{500} du} = \frac{u^{1500} \; u^{500}}{\int_0^1 u^{1500} (1-u)^{500} du} \quad , 0 < u < 1$$