Fall 2008, Applied Probability

1. a) $G_N(z) = \mathbb{E}[z^N] = \sum_{n=0}^{\infty} z^n e^{-\lambda} \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda z)^n}{n!} = e^{-\lambda} e^{e^{\lambda z}} = e^{\lambda(z-1)}$.

b) $M_X(t) = \mathbb{E}[e^{tX}] = \sum_{k=1}^{4} e^{tk} \frac{1}{4} = \frac{e^t + e^{2t} + e^{3t} + e^{4t}}{4}$

c) $\mathbb{E}[e^{tS}] = \mathbb{E}[\mathbb{E}[e^{tS} | N]]$

$$\mathbb{E}[e^{tS} | N=n] = \mathbb{E}[e^{t(X_1+\cdots+X_n)} | N=n] = \prod_{i=1}^{n} \mathbb{E}[e^{tX_i}] = \left(\frac{e^t + e^{2t} + e^{3t} + e^{4t}}{4}\right)^n$$

So, $\mathbb{E}[e^{tS} | N] = \left(\frac{e^t + e^{2t} + e^{3t} + e^{4t}}{4}\right)^N$ and thus,

$$M_S(t) = \mathbb{E}[e^{tS}] = \mathbb{E}\left[\left(\frac{e^t + e^{2t} + e^{3t} + e^{4t}}{4}\right)^N\right] = e^{\lambda\left(\frac{e^t + e^{2t} + e^{3t} + e^{4t}}{4} - 1\right)}$$

2. Since we are going to take limits, we can think that tickets are drawn with replacement. Because for large n the two cases give very similar results in means of probabilities.

a) $P(\text{at least one winning ticket}) = 1 - P(\text{no winning ticket})$

$$= 1 - \left(\frac{n^2 - n}{n^2}\right)^{2n}$$

$$= 1 - \left(1 - \frac{1}{n}\right)^{2n}$$

$\lim_{n\to\infty} P(\text{at least one winning ticket}) = 1 - \lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{2n} = 1 - e^{-2}$

b) Now, we can think that (since we assume "with replacement"), number of chosen winning tickets is a binomial r.v. There are $2n$ trials and the success probability is $\frac{n}{n^2} = \frac{1}{n}$. So,

$$P(\text{exactly 3 winning tickets}) = \binom{2n}{3}\left(\frac{1}{n}\right)^3\left(\frac{n-1}{n}\right)^{2n-3}$$

$$= \underbrace{\frac{2n(2n-1)(2n-2)}{6} \cdot \frac{1}{n^3}}_{\to \frac{4}{3}} \cdot \underbrace{\left(1 - \frac{1}{n}\right)^{2n}}_{\to e^{-2}} \underbrace{\left(\frac{n}{n-1}\right)^3}_{\to 1} \quad \text{as } n \to \infty$$

So, $\lim_{n\to\infty} P(\text{exactly 3 winning tickets}) = \frac{4e^{-2}}{3}$.

3. a) $P(rs < s') = \int_0^\infty \int_{rs}^\infty e^{-s} e^{-s'} \, ds' \, ds = \int_0^\infty e^{-s} \left[-e^{-s'} \right]_{rs}^\infty ds = \int_0^\infty e^{-s(r+1)} ds$

$= \left. \frac{e^{-s(r+1)}}{-(r+1)} \right|_0^\infty = -\frac{1}{r+1}(0-1) = \frac{1}{r+1}$

b) $D_1 = \frac{X_2 - X_1}{2}$, $D_2 = \frac{Y_2 - Y_1}{2}$, $D_3 = X_3 - \frac{X_1 + X_2}{2}$, $D_4 = Y_3 - \frac{Y_1 + Y_2}{2}$

$\rightarrow$ $Var(D_1) = \frac{1}{4}(Var X_2 + Var X_1) = \frac{1}{2}$ and $Var(D_2) = \frac{1}{2}$ (id)

$\rightarrow$ $Var(D_3) = 1 + \frac{1}{4}(1+1) = \frac{3}{2}$ and $Var(D_4) = \frac{3}{2}$ (id)

$\rightarrow$ $Cov(D_1, D_2) = Cov(D_1, D_4) = Cov(D_2, D_3) = Cov(D_3, D_4) = 0$ (independence)

$\rightarrow$ $Cov(D_1, D_3) = \frac{1}{2}\left[Cov\left(X_2, X_3 - \frac{X_1+X_2}{2}\right) - Cov\left(X_1, X_3 - \frac{X_1+X_2}{2}\right) \right]$

$= \frac{1}{2}\left[-\frac{1}{2} Cov(X_2, X_2) + \frac{1}{2} Cov(X_1, X_1) \right]$

$= 0$

$Cov(D_2, D_4) = 0$ (id, symmetry)

So, covariance of (D_1, D_2, D_3, D_4) is

$$\Sigma = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & 3/2 \end{bmatrix}$$

c) We have $A(X_1, Y_1)$, $B(X_2, Y_2)$ and $C(X_3, Y_3)$. The center of the circle is $\left(\frac{X_1+X_2}{2}, \frac{Y_1+Y_2}{2}\right)$ and its radius is $\sqrt{\left(X_2 - \frac{X_1+X_2}{2}\right)^2 + \left(Y_2 - \frac{Y_1+Y_2}{2}\right)^2}$ $= \sqrt{D_1^2 + D_2^2}$. So, we want to find probability that the square of the distance between C and center of the circle is less than $D_1^2 + D_2^2$.

The distance from C to center is $\sqrt{\left(X_3 - \frac{X_1+X_2}{2}\right)^2 + \left(Y_3 - \frac{Y_1+Y_2}{2}\right)^2} = \sqrt{D_3^2 + D_4^2}$

So, we want to find $P(D_3^2 + D_4^2 < D_1^2 + D_2^2)$.

Because of the matrix that we found in part (b), we know that

D_1, D_2, D_3, D_4 are independent normal random variables. Remember that if we have uncorrelated multivariate normal random variables then they are independent. So, there exists U_1, U_2, U_3, U_4 iid $N(0,1)$ r.v. so that $D_1 = \frac{1}{\sqrt{2}} U_1$, $D_2 = \frac{1}{\sqrt{2}} U_2$, $D_3 = \sqrt{\frac{3}{2}} U_3$ and $D_4 = \sqrt{\frac{3}{2}} U_4$. So,

$$P(D_3^2 + D_4^2 < D_1^2 + D_2^2) = P\left(\frac{3}{2} U_3^2 + \frac{3}{2} U_4^2 < \frac{1}{2} U_1^2 + \frac{1}{2} U_2^2\right)$$

$$= P(3(U_3^2 + U_4^2) < U_1^2 + U_2^2)$$

Now, we know $U_i^2 \sim \chi_1^2$ for $i = 1, \ldots, 4$. Since they are all independent. $U_3^2 + U_4^2 \sim \chi_2^2 = \text{Exponential}(2)$ and $U_1^2 + U_2^2 \sim \chi_2^2 = \text{Exponential}(2)$.

So, letting $U_3^2 + U_4^2 =: S_1$ and $U_1^2 + U_2^2 =: S_2$, we have

$$P(D_3^2 + D_4^2 < D_1^2 + D_2^2) = P(3S_1 < S_2)$$

where S_1 and S_2 are iid Exponential (2)

Now, again for $r > 0$ and $M, N \sim \text{Exponential}(\lambda)$,

$$P(rM < N) = \int_0^\infty \int_{rm}^\infty \frac{1}{\lambda^2} e^{-\frac{m}{\lambda}} e^{-\frac{n}{\lambda}} \, dn \, dm = \int_0^\infty \frac{1}{\lambda} e^{-\frac{m}{\lambda}} \left[-e^{-\frac{n}{\lambda}}\right]_{rm}^\infty dm$$

$$= \int_0^\infty \frac{1}{\lambda} e^{-\frac{m}{\lambda}(r+1)} \, dm = \left[\frac{e^{-\frac{m}{\lambda}(r+1)}}{-(r+1)}\right]_0^\infty = \frac{1}{r+1}$$

Thus, since S_1, S_2 are iid $\sim$ Exponential (2), we have (with $r = 3$)

$$P(D_3^2 + D_4^2 < D_1^2 + D_2^2) = P(3S_1 < S_2) = \frac{1}{4}$$