Fall 2007, Applied Probability

1. Let A_i be the event that i^{th} couple survives. Then define $N = \sum_{i=1}^{n} 1_{A_i}$ to be the number of surviving couples.

If $m > 2n-2$, then $\mathbb{E}[N] = 0$. So, suppose $m \leq 2n-2$. Then,

$$\mathbb{E}[N] = \sum_{i=1}^{n} P(A_i)$$

$$P(A_i) = \frac{\binom{2n-2}{m}}{\binom{2n}{m}} = \frac{(2n-2)(2n-3)\cdots(2n-2-m+1)}{\cancel{m!}} \; \frac{\cancel{m!}}{2n(2n-1)\cdots(2n-m+1)}$$

$$= \frac{\cancel{(2n-2)}\cancel{(2n-3)}\cdots\cancel{(2n-m+1)}(2n-m)(2n-m-1)}{2n(2n-1)\cancel{(2n-2)}\cdots\cancel{(2n-m+1)}}$$

$$= \frac{(2n-m)(2n-m-1)}{2n(2n-1)}$$

So, $\mathbb{E}[N] = \sum_{i=1}^{n} \frac{(2n-m)(2n-m-1)}{2n(2n-1)} = \frac{(2n-m)(2n-m-1)}{2(2n-1)}$

2. a) $F_Z(z) = P(Z \leq z) = P(XY \leq z) = P(XY \leq z \mid Y=1)P(Y=1) + P(XY \leq z \mid Y=-1)P(Y=-1)$

$$= P(X \leq z \mid Y=1) * \frac{1}{2} + P(-X \leq z) * \frac{1}{2} = \frac{1}{2}\left(P(X \leq z) + P(X \geq -z)\right)$$

$$= \frac{1}{2}\left(P(X \leq z) + P(X \leq z)\right) = P(X \leq z) = F_X(z)$$

Since $F_Z(z) = F_X(z)$ $\forall z \in \mathbb{R}$, we deduce that Z is also normally distributed.

b) $\mathrm{Cov}(X, Z) = \mathbb{E}[XZ] - \mathbb{E}[X]\mathbb{E}[Z]$

$$= \mathbb{E}[X^2 Y] - \mathbb{E}[X]\mathbb{E}[XY]$$

$$= \mathbb{E}[X^2]\mathbb{E}[Y] - \mathbb{E}[X]^2\mathbb{E}[Y]$$

independence of X and Y

$$= \mathbb{E}[Y]\,\mathrm{Var}(X)$$

$$= \left(\tfrac{1}{2}(-1) + \tfrac{1}{2}(1)\right) * 1$$

$$= 0$$

Thus, X and Z are uncorrelated.

c) If X and Z have bivariate normal distribution, for any region or in particular rectangle $R \subset \mathbb{R}^2$, we must have $P((X,Z) \in R) > 0$. So, consider

$$P((X,Z) \in [0,1] \times [1,2]) = P(X \in [0,1],\, Z \in [1,2])$$

$$= P(X \in [0,1],\ XY \in [1,2])$$

$$= P(X \in [0,1],\ XY \in [1,2] \mid Y=1)P(Y=1) + P(X \in [0,1],\ XY \in [1,2] \mid Y=-1)P(Y=-$$

by independence of X and Y

$$= \underbrace{P(X \in [0,1],\ X \in [1,2])}_{=0} * \frac{1}{2} + \underbrace{P(X \in [0,1],\ -X \in [1,2])}_{=0} * \frac{1}{2}$$

$$= 0$$

Thus, X and Y does not have bivariate normal distribution

3 Firstly observe that $\mathbb{E}[X_1 + \cdots + X_n] = 0 \quad \forall n$ Then, for $\varepsilon > 0$,

$$P\left(\left|\frac{X_1 + \cdots + X_n}{n} - 0\right| \geq \varepsilon\right) = P(|X_1 + \cdots + X_n| \geq n\varepsilon)$$

$$\leq \frac{\mathbb{E}[|X_1 + \cdots + X_n|^2]}{n^2 \varepsilon^2}$$ Chebychev's Inequality.

$$= \frac{1}{n^2 \varepsilon^2} \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right)$$

$$\rightarrow \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{1 \leq i < j \leq n} \mathrm{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} \phi(0) + 2 \sum_{1 \leq i < j \leq n} \phi(j-i)$$

$$= n\phi(0) + 2\left[(n-1)\phi(1) + (n-2)\phi(2) + \cdots + 2\phi(n-2) + \phi(n-1)\right]$$

$$= n\phi(0) + 2 \sum_{j=1}^{n-1} (n-j)\phi(j)$$

So, $$P\left(\left|\frac{X_1 + \cdots + X_n}{n} - 0\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2}\left(\frac{\phi(0)}{n} + \frac{2}{n^2} \sum_{j=1}^{n-1} (n-j)\phi(j)\right)$$

4. a) $G_{n+1}(s) = \mathbb{E}[s^{Z_{n+1}}]$

$$\rightarrow \mathbb{E}[s^{Z_{n+1}} \mid Z_n = n] = \mathbb{E}[s^{I_{n+1} + X_1 + X_2 + \dots + X_n} \mid Z_n = n]$$

$$= \mathbb{E}[s^{I_{n+1}}] * \prod_{i=1}^{n} \mathbb{E}[s^{X_i}]$$ by independence

$$= (sp + (1-p))[G(s)]^n$$

So, $\mathbb{E}[s^{Z_{n+1}} \mid Z_n] = (sp + (1-p))[G(s)]^{Z_n}$ and then

$$G_{n+1}(s) = \mathbb{E}[\mathbb{E}[s^{Z_{n+1}} \mid Z_n]]$$

$$= \mathbb{E}[(sp + (1-p))[G(s)]^{Z_n}]$$

$$= (sp + (1-p))\,\mathbb{E}[G(s)^{Z_n}]$$

$$= (sp + (1-p))\,G_n(G(s))$$

b) We know that for any r.v. X with probability generating function $G(s)$, $G'(1) = \mathbb{E}[X]$, and $G(1) = 1$. So,

$$G_{n+1}'(s) = p\,G_n(G(s)) + (sp + (1-p))\,G_n'(G(s))\,G'(s)$$

$$\Rightarrow \mu_{n+1} = G_{n+1}'(1) = p\,G_n(1) + G_n'(1)\,G'(1) = p + \mu_n \cdot \mu$$

Thus, $\mu_{n+1} = p + \mu_n \cdot \mu$

c) Suppose $\lim_{n\to\infty} \mu_n = \alpha$ then taking limit of both sides in $\mu_{n+1} = p + \mu_n \cdot \mu$, we get $\alpha = p + \alpha\mu \Rightarrow \alpha(1-\mu) = p \Rightarrow \alpha = \frac{p}{1-\mu}$. Thus

$$\lim_{n\to\infty} \mu_n = \frac{p}{1-\mu}$$