Fall 2006, Applied Probability

1. a) Let $U = X^2 + Y^2$ and $V = \frac{X^2}{X^2+Y^2}$ We see that $U > 0$ and $0 < V < 1$

Consider

$$J = \begin{vmatrix} 2X & 2Y \\ \frac{2XY^2}{(X^2+Y^2)^2} & -\frac{2X^2Y}{(X^2+Y^2)^2} \end{vmatrix} = -\frac{4X^3Y + 4XY^3}{(X^2+Y^2)^2} = \frac{-4XY(X^2+Y^2)}{(X^2+Y^2)^2}$$

$$= \frac{-4XY}{X^2+Y^2}$$

$\Rightarrow X^2 = UV$ and $Y^2 = (1-V)U$ $\Rightarrow$ $X = \pm\sqrt{UV}$ and $Y = \pm\sqrt{(1-V)U}$

Then since $f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$, $-\infty < x < \infty$ and $-\infty < y < \infty$), we have

$$f_{U,V}(u,v) = \frac{1}{2\pi} e^{-\frac{1}{2}u} \frac{u}{4u\sqrt{v(1-v)}}, u>0, \quad 0<v<1$$

$$= \frac{1}{8\pi} e^{-\frac{1}{2}u} \frac{1}{\sqrt{v(1-v)}}, u>0, \quad 0<v<1$$

From their joint density, we see that U and V are independent Thus, U and $\sqrt{V}$ are also independent, meaning that X^2+Y^2 and $\frac{X}{\sqrt{X^2+Y^2}}$ are independent.

b) Consider the r.v. $\frac{X+YZ}{\sqrt{1+Z^2}} \Big| Z$. Since X, Y and Z are iid $N(0,1)$, given Z, $\frac{X+YZ}{\sqrt{1+Z^2}}$ has normal distribution. Let us find its mean and variance

$$\mathbb{E}\left[\frac{X+YZ}{\sqrt{1+Z^2}} \Big| Z\right] = \frac{1}{\sqrt{1+Z^2}} \mathbb{E}[X|Z] + \frac{Z}{\sqrt{1+Z^2}} \mathbb{E}[Y|Z] = \frac{1}{\sqrt{1+Z^2}} \mathbb{E}[X] + \frac{Z}{\sqrt{1+Z^2}} \mathbb{E}[Y] = 0$$

$$\mathbb{E}\left[\left(\frac{X+YZ}{\sqrt{1+Z^2}}\right)^2 \Big| Z\right] = \frac{1}{1+Z^2}\left(\mathbb{E}[X^2|Z] + 2Z\,\mathbb{E}[XY|Z] + Z^2\,\mathbb{E}[Y^2|Z]\right)$$

$$= \frac{1}{1+Z^2}\left(\mathbb{E}[X^2] + 2Z\,\mathbb{E}[X]\,\mathbb{E}[Y] + Z^2\,\mathbb{E}[Y^2]\right)$$

$$= \frac{1}{1+Z^2}(1+Z^2)$$

$$= 1$$

$$Var\left(\frac{X+YZ}{\sqrt{1+Z^2}} \Big| Z\right) = \mathbb{E}\left[\left(\frac{X+YZ}{\sqrt{1+Z^2}}\right)^2 \Big| Z\right] - \mathbb{E}\left[\frac{X+YZ}{\sqrt{1+Z^2}} \Big| Z\right]^2 = 1-0 = 1$$

Thus, $\frac{X+YZ}{\sqrt{1+Z^2}} \big| Z \sim N(0,1)$. Letting $A := \frac{X+YZ}{\sqrt{1+Z^2}}$, consider

$$f_{A,Z}(a,z) = f_{A|Z}(a|z) f_Z(z)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-a^2/2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < a < \infty, \quad -\infty < z < \infty$$

So, A and Z are independent. Thus, $\frac{X+YZ}{\sqrt{1+Z^2}} \sim N(0,1)$

2. Let X be the event that there is no path from A to C.

a) We want to find $P(X^c)$. Now, consider that

$$P(X) = P(X \mid BD \text{ open}) P(BD \text{ open}) + P(X \mid BD \text{ closed}) P(BD \text{ closed})$$

$$= (1-p) P(X|BD \text{ open}) + p\, P(X|BD \text{ closed})$$

$\rightarrow P(X|BD \text{ open}) = pq(1-p^2) + q^2(1-p^2) + qp(1-p^2) = (1-p^2)^2$

$\rightarrow P(X|BD \text{ closed}) = p^2q^2 + 2pq^3 + q^4 + 2pq^3 + p^2q^2 = 2pq^2(p+2q) + q^4$

So, $P(X) = (1-p)(1-p^2)^2 + 2p^2q^2(p+2q) + pq^4$.

$$= q^3(1+p)^2 + 2p^2q^2(p+2q) + pq^4$$

$$= q^3 + 2pq^3 + p^2q^3 + 2p^3q^2 + 4p^2q^3 + pq^4$$

$$= 2p^3q^2 + (1+2p+5p^2)q^3 + pq^4$$

where $q = 1-p$. Thus $P(X^c) = 1 - 2p^3q^2 - (1+2p+5p^2)q^3 - pq^4$

b) $P(BD \text{ closed} \mid X^c) = \frac{P(BD \text{ closed}, X^c)}{P(X^c)} = \frac{P(BD \text{ closed}) - P(BD \text{ closed}, X)}{P(X^c)}$

$$= \frac{p - 2p^2q^2(p+2q) - pq^4}{1 - 2p^3q^2 - (1+2p+5p^2)q^3 - pq^4}$$