Fall 2005, Applied Probability

1. a) $a = \mathbb{E}[X_1] = P(X_1 = 1) = 1 - P(X_1 = 0) = 1 - P(\text{no one choses } 1^{st} \text{ floor})$

$$= 1 - \left(\frac{k-1}{k}\right)^m$$

b) $b = \mathbb{E}[X_1 X_2] = P(X_1 = 1, X_2 = 1)$

$\rightarrow P(X_1 = 1, X_2 = 1) = 1 - P(X_1 = 0, X_2 = 1) - P(X_1 = 1, X_2 = 0) - P(X_1 = 0, X_2 = 0)$

$* \; P(X_1 = 0, X_2 = 1) = P(X_1 = 0) - P(X_1 = 0, X_2 = 0)$

$$= \left(\frac{k-1}{k}\right)^m - \left(\frac{k-2}{k}\right)^m$$

Note that $P(X_1 = 1, X_2 = 0) = P(X_1 = 0, X_2 = 1)$ by symmetry.

$* \; P(X_1 = 0, X_2 = 0) = \left(\frac{k-2}{k}\right)^m$

So, $b = P(X_1 = 1, X_2 = 1) = 1 - 2\left(\frac{k-1}{k}\right)^m - 2\left(\frac{k-2}{k}\right)^m - \left(\frac{k-2}{k}\right)^m$

$$= 1 - 2\left(\frac{k-1}{k}\right)^m - 3\left(\frac{k-2}{k}\right)^m$$

c) $c = \mathrm{Cov}(X_1, X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]$

$$= 1 - 2\left(\frac{k-1}{k}\right)^m - 3\left(\frac{k-2}{k}\right)^m - \left[1 - \left(\frac{k-1}{k}\right)^m\right]^2$$

$$= -\left(\frac{k-1}{k}\right)^m \left[3 + \left(\frac{k-1}{k}\right)^m\right] = b - a^2$$

d) $\mathbb{E}[S] = \sum_{i=1}^{k} \mathbb{E}[X_i] = \sum_{i=1}^{k} P(X_i = 1) = k - k\,\frac{(k-1)^m}{k^m} = k - \frac{(k-1)^m}{k^{m-1}}$.

e) $\mathrm{Var}(S) = \sum_{i=1}^{k} \mathrm{Var}(X_i) + \sum_{1 \le i < j \le k} \mathrm{Cov}(X_i, X_j)$

$$= \sum_{i=1}^{k} (a - a^2) + \sum_{1 \le i < j \le k} (b - a^2)$$

$$= k(a - a^2) + \frac{k(k-1)}{2}(b - a^2)$$

2. a) We see that

$$X_n' = \frac{X_n + Y_n}{2} = \begin{cases} 1 & \text{wp } 1/4 \\ 0 & \text{wp } 1/2 \\ -1 & \text{wp } 1/4 \end{cases} \quad \text{and} \quad Y_n' = \frac{X_n - Y_n}{2} = \begin{cases} 1 & \text{wp } 1/4 \\ 0 & \text{wp } 1/2 \\ -1 & \text{wp } 1/4 \end{cases}$$

But they cannot take some of the values at the same time.

$P(X_n'=1, Y_n'=1) = P(X_n+Y_n=2, X_n-Y_n=2) = 0$

$P(X_n'=1, Y_n'=0) = P(X_n+Y_n=2, X_n-Y_n=0) = 1/4$

$P(X_n'=1, Y_n'=-1) = P(X_n+Y_n=2, X_n-Y_n=-2) = 0$

$P(X_n'=0, Y_n'=1) = P(X_n+Y_n=0, X_n-Y_n=2) = 1/4$

$P(X_n'=0, Y_n'=0) = P(X_n+Y_n=0, X_n-Y_n=0) = 0$

$P(X_n'=0, Y_n'=-1) = P(X_n+Y_n=0, X_n-Y_n=-2) = 1/4$

$P(X_n'=-1, Y_n'=1) = P(X_n+Y_n=-2, X_n-Y_n=2) = 0$

$P(X_n'=-1, Y_n'=0) = P(X_n+Y_n=-2, X_n-Y_n=0) = 1/4$

$P(X_n'=-1, Y_n'=-1) = P(X_n+Y_n=-2, X_n-Y_n=-2) = 0$

Thus, we have

$$Z_n = \begin{cases} (1,0) & \text{wp } 1/4 \\ (0,1) & \text{wp } 1/4 \\ (0,-1) & \text{wp } 1/4 \\ (-1,0) & \text{wp } 1/4 \end{cases}$$

and since $(X_n)_{n\geq 0}$, $(Y_n)_{n\geq 0}$ are independent, $(Z_n)_{\geq 0}$ is also independent.
Thus $(Z_n)_{n\geq 0}$ is a simple random walk on $\mathbb{Z}^2$.

b) From above definition of Z_n, for any standard basis vector e_i in $\mathbb{R}^2$ (which are $(0,1),(1,0)$) we have $P(Z_1=e_i)=1/4$.

c) From above calculations, $P(Z_n=0) = P(X_n+Y_n=0, X_n-Y_n=0) = P(X_n=0, Y_n=0) = 0$.

d) Assuming $Z_n' = \sum_{k=1}^{n} Z_k$, consider

$$P(Z_{2n}'=0) = P\left(\sum_{k=1}^{2n} Z_k = 0\right) = P\left(\sum_{k=1}^{2n} X_k'=0, \sum_{k=1}^{2n} Y_k'=0\right)$$

$$= P\left(\sum_{k=1}^{2n} X_k + \sum_{k=1}^{2n} Y_k = 0, \sum_{k=1}^{2n} X_k - \sum_{k=1}^{2n} Y_k = 0\right) = P\left(\sum_{k=1}^{2n} X_k=0, \sum_{k=1}^{2n} Y_k=0\right)$$

$$\underset{\substack{\uparrow \\ \text{by independence}}}{=} P\left(\sum_{k=1}^{2n} X_k=0\right) P\left(\sum_{k=1}^{2n} Y_k=0\right) \underset{\substack{\uparrow \\ \text{identically}}}{=} P\left(\sum_{k=1}^{2n} X_k=0\right)^2$$

4. Firstly let $U \sim$ Uniform$[-a, a]$, let us find mean and variance of $|U|$

$$\mathbb{E}[|U|] = \int_{-a}^{a} |u| \frac{1}{2a} du = \frac{1}{2a}\left[\int_{-a}^{0} -u du + \int_{0}^{a} u du\right] = \frac{1}{2a} \times 2 \int_{0}^{a} u du = \frac{1}{a}\left[\frac{u^2}{2}\right]_0^a$$

$$= \frac{a}{2}$$

$$\mathbb{E}[|U|^2] = \mathbb{E}[U^2] = Var(U) + \mathbb{E}[U]^2 = \frac{(a+a)^2}{12} + 0 = \frac{a^2}{3}$$

So, $Var(|U|) = \frac{a^2}{3} - \frac{a^2}{4} = \frac{a^2}{12}$

So, we have $\mathbb{E}\left[\sum_{i=1}^{n} |X_i|\right] = \sum_{i=1}^{n} \frac{1}{4} 10^{-2} = \frac{n}{4} 10^{-2}$ and

$Var\left(\sum_{i=1}^{n} |X_i|\right) = \sum_{i=1}^{n} \frac{1}{48} 10^{-4} = \frac{n}{48} 10^{-4}$. Then by CLT, we know

$$\frac{\sum_{i=1}^{n} |X_i| - \frac{n}{4} 10^{-2}}{\sqrt{\frac{n}{48}}\, 10^{-2}} \xrightarrow{d} N(0,1)$$

So,

$$P\left(\sum_{i=1}^{n} |X_i| < \lambda\right) = P\left(\frac{\sum_{i=1}^{n} |X_i| - \frac{n}{4} 10^{-2}}{\sqrt{\frac{n}{48}}\, 10^{-2}} < \frac{\lambda - \frac{n}{4} 10^{-2}}{\sqrt{\frac{n}{48}}\, 10^{-2}}\right)$$

$$\approx P\left(Z < \frac{\lambda - \frac{n}{4} 10^{-2}}{\sqrt{\frac{n}{48}}\, 10^{-2}}\right)$$

where $Z \sim N(0,1)$. We want to find λ so that

$$P\left(Z < \frac{\lambda - \frac{n}{4} 10^{-2}}{\sqrt{\frac{n}{48}}\, 10^{-2}}\right) \approx 0.99$$

Since $Z \sim N(0,1)$, from table, we get $\frac{\lambda - \frac{n}{4} 10^{-2}}{\sqrt{\frac{n}{48}}\, 10^{-2}} \approx 2.325$ and so,

$$\lambda \approx 0.2325 \frac{\sqrt{n}}{4\sqrt{3}} + n\, 0.0025$$

$$P\left(\sum_{k=1}^{2n} X_k = 0\right) = \binom{2n}{n}\frac{1}{2^n}\frac{1}{2^n} = \binom{2n}{n}\frac{1}{2^{2n}}$$

Then, by Stirling's formula,

$$P(Z'_{2n}=0) = \binom{2n}{n}^2 \frac{1}{2^{4n}} \approx \left[\frac{(2n)^{2n}\cancel{e^{-2n}}\sqrt{4\pi n}}{n^{2n}\cancel{e^{-2n}}\,2\pi n}\right]^2 \frac{1}{2^{4n}} = \frac{2^{4n}}{\pi n}\,\frac{1}{2^{4n}} = \frac{1}{\pi n}$$

Thus, $P(Z'_{2n}=0) \longrightarrow 0$ as $n \to \infty$

3. By definition $G_X(s) = \mathbb{E}[s^X]$. Firstly consider that

$$E[s^X \mid U=u] = \sum_{k=0}^{n} s^k P(X=k \mid U=u)$$

$$= \sum_{k=0}^{n} s^k \binom{n}{k} u^k (1-u)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (su)^k (1-u)^{n-k}$$

$$= (1+u(s-1))^n$$

So, $E[s^X|U] = (1+U(s-1))^n$. Then,

$$\mathbb{E}[\mathbb{E}[s^X|U]] = \mathbb{E}[(1+U(s-1))^n]$$

$$= \int_0^1 (1+u(s-1))^n du$$

$$= \left[\frac{(1+u(s-1))^{n+1}}{(n+1)(s-1)}\right]_0^1$$

$$= \frac{s^{n+1}-1}{(n+1)(s-1)}$$

So, we have $G_X(s) = \mathbb{E}[s^X] = \mathbb{E}[\mathbb{E}[s^X|U]] = \frac{1}{n+1}\,\frac{s^{n+1}-1}{s-1}$

On the other hand let Y be a uniformly distributed r.v. on $\{0,1,\ldots,n\}$
The probability generating function of Y is

$$G_Y(s) = \mathbb{E}[s^Y] = \sum_{k=0}^{n} s^k P\{Y=k\} = \frac{1}{n+1}\sum_{k=0}^{n} s^k = \frac{1}{n+1}\,\frac{s^{n+1}-1}{s-1}$$

Since $G_X(s) = G_Y(s)$, X and Y have the same distribution. That is, X is uniformly distributed on $\{0,1,\ldots,n\}$.