MATH 505a QUALIFYING EXAM September 13, 2005

Last Name: _____ **First Name:** _____

ID#: _____ **Signature:** _____

1. A building has $k > 1$ floors above the street level.

Some number $m > 1$ of people enter at street level and board an elevator. They choose floors independently, uniformly at random. Let X_i be the indicator of the event that at least one person selects floor i, so that $S = X_1 + \ldots + X_k$ is the number of stops the elevator must make.

a) Find the constant $a = \mathbf{E}X_1$ as a function of $m,\ k$.

b) Find the constant $b = \mathbf{E}X_1X_2$ as a function of m, k.

c) Find $c = \mathrm{Cov}(X_1, X_2)$ in terms of a, b.

d) Use the method of indicators to find $\mathbf{E}S$,

e) Use the variance-covariance expansion to find $\mathrm{Var}(S)$ in terms of a, b, m.

2. Let $(X_n, n \geq 0)$ and $(Y_n, n \geq 0)$ be two independent simple random walks on $\mathbb{Z}$ starting at zero.

a) Prove that the sequence of ordered pairs $(Z_n = (X'_n, Y'_n), n \geq 0)$, where

$$X'_n = \frac{X_n + Y_n}{2} \text{ and } Y'_n = \frac{X_n - Y_n}{2},$$

is a simple random walk on $\mathbb{Z}^2$.

b) For a standard basis vector $\mathbf{e}_i$ in $\mathbf{R}^2$, find $\mathbf{P}(Z_1 = -\mathbf{e}_i)$.

c) Find $\mathbf{P}(Z_n = 0)$.

d) Find the asymptotic of $\mathbf{P}\left(Z'_{2n} = 0\right)$ as $n \longrightarrow \infty$.

Hint: Use Stirling's formula: $n! \sim n^n e^{-n}\sqrt{2\pi n}$.

3. Let X have the binomial distribution $\mathrm{Bin}(n, U)$, where U is uniform on $(0, 1)$. Show that X is uniformly distributed on $\{0, 1, \ldots, n\}$.

Hint. Compute the generating function of X.

4. The n randomly chosen real numbers $a_1, \ldots, a_n$ are rounded off to the nearest hundredth $a_1 + X_1, \ldots a_n + X_n$, where the round-off errors $X_1, \ldots, X_n$ are assumed to be independent and uniformly distributed on $[-\frac{1}{2}10^{-2}, \frac{1}{2}10^{-2}]$.

Use the CLT to find a number $\lambda > 0$ (depending upon n) such that

$$\mathbf{P}(\sum_{i=1}^{n} |X_i| < \lambda) \approx 0.99.$$

STANDARD NORMAL DISTRIBUTION FUNCTION

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt, \qquad x \geq 0.$$

For $x < 0$, use the relation $\Phi(x) = 1 - \Phi(-x)$.

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990