Fall 2004, Applied Probability

1. Let B_i be the event that first person chooses i^{th} coin and A_i be the event that second person wins the game by choosing i^{th} coin.

$\rightarrow P(A_2|B_1) = 3/9$, $P(A_3|B_1) = 1/3 + 2/3 * 1/3 = 5/9$

So, if first person chooses first coin, second one has greater probability of winning by choosing third coin.

$\rightarrow P(A_1|B_2) = 6/9$, $P(A_3|B_2) = 3/9$

If first person chooses second coin, second person has greater probability of winning by choosing second coin.

$\rightarrow P(A_1|B_3) = 2/3 * 2/3 = 4/9$, $P(A_2|B_3) = 6/9$

If first person chooses third coin, second one has greater probability of winning by choosing second coin.

So, with the given probabilities and coins, there is no superior coin that makes the person who choses it the winner all the time. In this case, being second person to choose a coin is better, since the second person can make his/her choice to increase the probability of winning.

2. same as #1 of Spring 2006.

3. Firstly, we can see that $Y(y)$ is a nonnegative integer valued r.v. Then, consider that for $n \in \mathbb{N}$,

$$P(Y(y)=n) = P(X_1 \le y, X_2 \le y, \dots, X_{n-1} \le y, X_n > y)$$

$$= [F(y)]^{n-1}(1-F(y))$$) by independence.

We see that $Y(y) \sim \text{Geometric}(1-F(y))$. So, $E[Y(y)] = \frac{1}{1-F(y)}$. Then,

$$P(Y(y) \le E[Y(y)]) = P\left(Y(y) \le \frac{1}{1-F(y)}\right)$$

$$= \sum_{k=1}^{\left\lfloor \frac{1}{1-F(y)} \right\rfloor} [F(y)]^{k-1}(1-F(y))$$

$$= (1-F(y))\frac{1-[F(y)]^{\left\lfloor \frac{1}{1-F(y)} \right\rfloor}}{1-F(y)}$$

$$= 1-[F(y)]^{\left\lfloor \frac{1}{1-F(y)} \right\rfloor}$$ is the desired probability.

Now consider

$$\lim_{y\to\infty} P(Y(y) \leq \mathbb{E}[Y(y)]) = 1 - \lim_{y\to\infty} F(y)^{\left\lfloor \frac{1}{1-F(y)} \right\rfloor}$$

$$= 1-\exp\left\{\lim_{y\to\infty}\left(\left\lfloor \frac{1}{1-F(y)}\right\rfloor \ln F(y)\right)\right\}$$

$$= 1-\exp\left\{\lim_{y\to\infty}\left(\frac{1}{1-F(y)} \ln F(y)\right)\right\}$$

L'Hôspital's Rule

$$= 1-\exp\left\{\lim_{y\to\infty}\frac{1/F(y) \times f(y)}{-f(y)}\right\}$$

$$= 1-\exp\left\{\lim_{y\to\infty} -\frac{1}{F(y)}\right\}$$

$$= 1-e^{-1}$$

4 a) Let X be a $N(0,1)$ r.v. Consider

$$M_Z(t) = \mathbb{E}[e^{tZ}] = \int_{-\infty}^{\infty} e^{tz}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz = e^{t^2/2}\underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-t)^2}\,dz}_{=1,\text{ since the integrand is pdf of } N(t,1) \text{ r.v.}} = e^{t^2/2}$$

Then, since $X_i = \mu + \sigma Z$, we have

$$M_{X_i}(t) = \mathbb{E}[e^{t\mu+\sigma t Z}] = e^{t\mu}\,\mathbb{E}[e^{(\sigma t)Z}] = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Then,

$$M_{S_n}(t) = \mathbb{E}[e^{tS_n}] = \mathbb{E}[e^{X_1 t + \cdots X_n t}] \overset{\text{by independence}}{=} \prod_{i=1}^{n}\mathbb{E}[e^{X_i t}] = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$= e^{n\mu t + \frac{n\sigma^2 t^2}{2}}$$

b) $\mathbb{E}[e^{S_n}] = M_{S_n}(1) = e^{n\mu + \frac{n\sigma^2}{2}}$.

c) Let us denote $\frac{S_n - n\mu}{\sqrt{n}\sigma}$ by T_n. Consider

$$M_{T_n}(t) = \mathbb{E}\left[e^{tT_n}\right]$$

$$= \mathbb{E}\left[e^{\frac{tS_n - tn\mu}{\sqrt{n}\sigma}}\right]$$

$$= e^{\frac{-tn\mu}{\sqrt{n}\sigma}} \cdot \mathbb{E}\left[e^{(t/\sqrt{n}\sigma)S_n}\right]$$

$$= e^{(-\sqrt{n}\mu/\sigma)t} \cdot e^{(\sqrt{n}\mu/\sigma)t} \, e^{t^2/2}$$

$$= e^{t^2/2}$$