Math 505a Qualifying Exam Problems
Fall 2003

Problem 1.

Let X and Y be discrete random variables taking values $0, 1, 2, ...$ Assume that the joint probability generating function

$$G_{X,Y}(t,s) = \frac{(1-[p_1+p_2])^n}{(1-[p_1 s + p_2 t])^n}$$

where $p_1 + p_2 \leq 1$ and n is a positive integer.

a. Find the marginal mass functions of X and Y.

b. Find the marginal mass function of $X + Y$.

c. Find the conditional probability generating function $G_{X|Y}(s|i) = E\left[s^X|Y=i\right]$

Problem 2.

Let $X_1, X_2, ...$ be iid with distribution function $F(x) = \sqrt{x}$ for $0 \leq x \leq 1$, and let $M_n = \min\{X_1, .., X_n\}$. Show that $n^2 M_n$ converges in distribution, and find the limiting distribution function.

Problem 3.

Consider gambler's ruin with fair bets. Player A starts with k dollars and player B with $N - k$ dollars. Each player wins each round with probability $1/2$, gaining \$1 from his opponent. The game ends when one player (the winner of the game) has all N dollars. Let W_k be the event that A wins the game, let R_k be the number of rounds played in the game, and for $1 \leq k \leq N$ let $a_k = E(R_k|W_k)$. It is standard, and you may take as given, that $P(W_k) = k/N$ for all $0 \leq k \leq N$.

(a) Show that

$$(k+1)a_{k+1} - 2ka_k + (k-1)a_{k-1} = -2k, \quad 1 \leq k \leq n. \qquad (1)$$

Note this requires that you choose a definition for $(k-1)a_{k-1}$ when $k = 1$.

(b) The general solution of (1) has form $a_k = b + ck + dk^2$, where b, c are arbitrary constants and d is a specific constant (you may take this fact as given.) Use this to find $E(R_k|W_k)$ explicitly.

Problem 4.

Let X and Y be independent normal random variables with zero means, and unit variances.

a. Prove that $U = (X+Y)/\sqrt{2}$ and $V = (X-Y)/\sqrt{2}$ are independent and Gaussian with zero means and unit variances.

b. Prove that $P(X+Y \leq z | X > 0, Y > 0) = P\left(|U| \leq z/\sqrt{2}, |V| \leq z/\sqrt{2}\right)$