Math 505a Qualifying Exam
Fall 2002

You should try at least 3 problems; you may try all 4.

Problem 1.

An inefficient secretary places n different letters into n differently addressed envelopes at random.

(i) Find the expected number of letters arriving at the proper destination.

(ii) Find the probability that at least one letter will arrive at the proper destination.

Problem 2. Let a_n be the probability that n Bernoulli trials with success probability p result in an even number of successes.

a) Find a relation between a_n and a_{n-1}.

b) Use (a) to calculate a_n.

Problem 3.

If U is uniform on $(0, 2\pi)$ and Z, independent of U, is exponential with mean 1, show that X and Y, defined by

$$X = \sqrt{2Z}\cos U, \quad Y = \sqrt{2Z}\sin U$$

are independent standard normal random variables.

Problem 4.

Let A_n, $n = 1, 2, \ldots$, be a sequence of events (not necessarily independent) such that

a) $\sum_{n=1}^{\infty} P(A_n) = \infty$ and

b) $P(A_n \cap A_m) \leq P(A_n)P(A_m) \quad \forall\, n \neq m.$

True or false: $P\{A_n \text{ i.o.}\} = 1$. Explain. HINT: Consider the mean and variance of $\sum_{i=1}^{n} I_{A_i}$, where I_A denotes the indicator function of the event A. Also, "i.o." means "infinitely often."