REAL ANALYSIS GRADUATE EXAM

Fall 2025

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall that a measure μ is *semifinite* if for all $E \in \mathcal{M}$ with $\mu(E) > 0$, there exists $A \in \mathcal{M}$, $A \subseteq E$, such that $0 < \mu(A) < \infty$. Prove that if μ is semifinite, then for every $E \in \mathcal{M}$, we have

$$\mu(E) = \sup\{\mu(A) : A \in \mathcal{M}, A \subseteq E, \mu(A) < \infty\}.$$

2. Compute the limit

$$\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\, dx$$

justifying your answer.

3. (i) Let $f \in \mathscr{L}^1([0,1], m)$, m the Lebesgue measure on $[0,1]$ and m^2 the Lebesgue measure on $[0,1] \times [0,1]$. For $\varepsilon > 0$ put

$$A_\varepsilon = \{(x,y) \in [0,1] \times [0,1] : |x - y| < \varepsilon\}.$$

Prove that

(i) $\iint\limits_{A_\varepsilon} |f(x) - f(y)|\, dm^2(x,y) \leq 4\varepsilon \|f\|_{\mathscr{L}^1([0,1],m)}$.

(ii) $\lim\limits_{\varepsilon\to 0} \frac{1}{\varepsilon} \iint\limits_{A_\varepsilon} |f(x) - f(y)|\, dm^2(x,y) = 0$. (Hint: Coordinate change?)

4. Let q and n be natural numbers with $q, n \geq 1$ and $q < n$. Assume that $E_1, E_2, \ldots, E_n$ are n Lebesgue measurable subsets of $[0,1]$ with the property that each point $x \in [0,1]$ is in at least q of the above subsets. Prove that there exists at least one E_j such that

$$m(E_j) \geq \frac{q}{n}$$

where m is the Lebesgue measure on $[0,1]$. (Hint: Counting function?)