PDE Screening Exam
Spring 2025

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Let $u \not\equiv 0$ be a $C^2(\mathbb{R}^n)$ function ($n > 1$). Define

$$m_x(r) = r^{1-n} \int_{\partial B(x,r)} u(y) dS(y).$$

   (a) Show that

$$\frac{dm_x(r)}{dr} = r^{1-n} \int_{B(x,r)} \Delta u(y) dy.$$

   (b) Let u solve $-\Delta u = \phi(u)$ for some continuous function ϕ. Assume that $u(x) \geq 1$ for all $x \in \mathbb{R}^n$, and that $\phi(\xi) \geq 0$ for all $\xi \geq 1$. Using the result of part (a) above, prove that if $u(x_0) = 1$ for some $x_0 \in \mathbb{R}^n$, then $u(x) \equiv 1$ for all $x \in \mathbb{R}^n$.

2. Use the method of characteristics to solve the following partial differential equation:

$$\begin{aligned} \partial_t u - u \partial_x u &= 3u, \quad x \in \mathbb{R}, \quad t > 0, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}. \end{aligned}$$

3. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded domain with smooth boundary. Assume that $u(t,x) \geq 0$ is a smooth function solving

$$\begin{cases} \partial_t u - \Delta u = -u^4 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

   with $u|_{t=0} = u_0$. Here ν is the unit normal to the boundary. Let

$$E(t) = \int_\Omega u^2(t,x) dx.$$

   Show that there exists a constant $C > 0$ such that for each $t > 0$

$$E(t) \leq \frac{1}{(E^{-\frac{3}{2}}(0) + Ct)^{\frac{2}{3}}}$$