Numerical Analysis Preliminary Examination Spring 2025

January 5, 2025

Problem 1.

(a) Let $M, N \in \mathbb{R}^{n\times n}$ be given by

$$M = \begin{pmatrix} R & S \\ 0 & T \end{pmatrix} \text{ and } N = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$$

with R, T, X, Z square and nonsingular. Show M and N are nonsingular by finding well-defined expressions for M^{-1} and N^{-1}.

(b) Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and have L-U decomposition $A = LU$ with $L_{ii} = 1$, $i = 1, 2, ..., n$. Set $A_1 = A$, $L_1 = L$, $U_1 = U$ and partition $L = L_1$ and $U = U_1$ as

$$L = L_1 = \begin{pmatrix} 1 & 0 \\ l_1 & L_2 \end{pmatrix} \text{ and } U = U_1 = \begin{pmatrix} u_{1,1} & u_1^T \\ 0 & U_2 \end{pmatrix}$$

where $L_2, U_2 \in \mathbb{R}^{(n-1)\times(n-1)}$, and $l_1, u_1 \in \mathbb{R}^{n-1}$. Find an expression for $A^{-1} = A_1^{-1}$ in terms of A_2^{-1} where $A_2 = L_2U_2 \in \mathbb{R}^{(n-1)\times(n-1)}$.

(c) Use your result in part (b) to design an $n-1$ step algorithm for computing $A^{-1} = A_1^{-1}$. Be certain to explicitly specify the algorithm's initial step. Specify your algorithm in the form of pseudo-code or either Matlab or Python code. (Hint: Your algorithm should step backwards from n to 1.)

Problem 2.

(a) Let $x, y \in \mathbb{R}^n$, with $\|x\|_2 = \|y\|_2$ and $x \neq y$. Let $v = x - y$, set $H = I - 2\frac{vv^T}{v^Tv} \in \mathbb{R}^{n\times n}$ and show that $Hx = y$.

(b) Let $x = [3, 4, 5]^T$ and $y = [5, 0, 5]^T$. What is H and show that it works.

(c) Let $v \in \mathbb{R}^n$ be given and set $H = I - 2\frac{vv^T}{v^Tv} \in \mathbb{R}^{n\times n}$. Find $u \in \mathbb{R}^n$ such that $H = I - uu^T$.

(d) With $n = 2$, let $u \in \mathbb{R}^2$ from part (c) be given by $u = [u_1, u_2]^T \in \mathbb{R}^2$ and show that $u_1^2 - 1 = 1 - u_2^2$, $0 \leq u_1^2, u_2^2 \leq 2$, and $-1 \leq u_1^2 - 1 = 1 - u_2^2 \leq 1$.

(e) For $u \in \mathbb{R}^2$ from part (c), use part (d) to show that there exists a $\theta \in [0, 2\pi]$ such that $cos\theta = u_1^2 - 1 = 1 - u_2^2$ and $-u_1 u_2 = sin\theta$.

(f) For the case $n = 2$ considered in parts (d) and (e), show that there exists a $\theta \in [0, 2\pi]$ such that $H = \begin{pmatrix} -cos\theta & sin\theta \\ sin\theta & cos\theta \end{pmatrix}$.

(g) For $x \in \mathbb{R}^2$ given, and H as in part (f), provide a geometric interpretation for $y = Hx$. (Hint: Let $x = [Rcos\phi, Rsin\phi]^T$, for some $R > 0$ and $\phi \in [0, 2\pi]$.)

Problem 3.

Recall, Francis' QR algorithm (to find all the eigenvalues of a matrix $A \in \mathbb{R}^{n \times n}$) is defined as follows.

Let $A_1 = A$.

For $k = 1, 2, \ldots$ until an appropriate convergence criterion is met:

- find a QR factorization of A_k: $A_k = Q_k R_k$ and
- let $A_{k+1} = R_k Q_k$.

In this problem we will show that this method converges under the following assumptions.

- We assume A has real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying

$$|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|.$$

- Let $D = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$ and let X be the matrix whose i'th column is an eigenvector corresponding to λ_i, so $AX = XD$. We assume X^{-1} has an LU decomposition

$$X^{-1} = LU$$

where the matrix L is lower triangular with 1's along its diagonal.

- For each $k \in \mathbb{N}$, we assume $Q_k R_k$ is the QR factorization of A_k where the matrix R_k has positive entries on the diagonal.

(a) Let $P_k = Q_1 Q_2 \ldots Q_k$ and $U_k = R_k R_{k-1} \ldots R_1$. Show

$$A_{k+1} = P_k^T A P_k$$
$$\text{and} \quad A^k = P_k U_k.$$

Notice, P_k is orthogonal and U_k is upper triangular with positive entries on the diagonal, so the second equation is the QR factorization of A^k where the diagonal entries of the upper triangular matrix are positive. *Hint: First show $A_{k+1} = P_k^T A P_k$. Then show $P_k U_k = P_{k-1} A_k U_{k-1}$ and use $P_k A_{k+1} = A P_k$.*

(b) Let $X = QR$ be a QR-factorization of X and $X^{-1} = LU$ an LU-factorization of X^{-1}. Show

$$A^k = QR(I + E_k)D^kU$$

where E_k is lower triangular and converges to 0 as $k \to \infty$.

(c) Comparing the two expressions for A^k in parts (a) and (b) show

$$\begin{aligned} P_k &= Q\tilde{Q}_k\tilde{D}_k \\ \text{and} \quad U_k &= \tilde{D}_k\tilde{R}_kRD^kU \end{aligned}$$

where $I + RE_kR^{-1} = \tilde{Q}_k\tilde{R}_k$ is a QR-factorization (where the diagonal entries in $\tilde{R}_k$ are all positive) and $\tilde{D}_k$ is a diagonal matrix such that $\tilde{D}_k^2 = I$.

(d) Using the expression for P_k in part (c) and the expression for A_{k+1} in part (a) show

$$A_{k+1} = \tilde{D}_k^T\tilde{Q}_k^TRDR^{-1}\tilde{Q}_k\tilde{D}_k$$

and use this to show A_{k+1} converges to an upper triangular matrix with diagonal elements λ_1, λ_2, …, λ_n in that order.

Problem 4.

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Follow the steps below to prove that the Gauss-Seidel method to solve $Ax = b$ converges for any arbitrary choice of the initial approximation $x^{(0)}$.

(a) Write $A = L + D + L^T$ for appropriate matrices L and D.

(b) Show that the Gauss-Seidel method can be written as $x^{(k+1)} = Bx^{(k)} + d$ and determine the matrix B and the vector d.

(c) Let u be an eigenvector of B and $-\lambda$ its corresponding eigenvalue. Show that

$$u^*Au = (1 + \lambda)u^*(L + D)u$$

where u^* denotes the conjugate transpose of u.

(d) Show that

$$\frac{1}{(1+\lambda)} + \frac{1}{(1+\overline{\lambda})} > 1$$

Hint: Take the conjugate transpose on both sides of the result in part (c).

(e) Show that the spectral radius of B is less than 1. Finally, use the fact (do not prove) that the iteration in part (b) converges for any initial approximation $x^{(0)}$ if and only if the spectral radius of B is less than one, to prove our result about the convergence of the Gauss-Seidel method.