PDE Screening Exam
Fall 2024

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Consider a linear transport equation ($a > 0$) with drag ($b > 0$):

$$u_t + au_x + bu = 0. \tag{1}$$

   (a) Find the solution $u(t, x)$ to Eq. (1) for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ with the initial condition $u(0, x) = f(x)$.

   (b) Find the solution $u(t, x)$ to Eq. (1) for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+$, subject to the initial condition $u(0, x) = f(x)$ and the boundary condition $u(t, 0) = g(t)$, where $f(0) = g(0)$.

2. Let $\Omega \subset \mathbb{R}^n$ ($n > 1$) be a bounded domain with smooth boundary. Assume that $u \in C^2(\bar{\Omega})$ solves

$$\begin{aligned} \Delta u &= u^7 + 2u^5 + 3u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

   Show that u is identically zero.

3. Let $\Omega \subset \mathbb{R}^n$ ($n > 1$) be a bounded domain with smooth boundary. Let $u \in C^2([0, \infty) \times \bar{\Omega})$, which solves the equation

$$u_{tt} - \Delta u = u$$

   with the boundary condition $u = 0$ on $\partial\Omega$. Let

$$E(t) = \frac{1}{2} \int_{\Omega} u_t^2(t, x) + |\nabla_x u(t, x)|^2 \, dx$$

   Prove that there exists $C > 0$ independent of t such that

$$E(t) \le \exp(Ct)E(0) \quad \text{for } t \ge 0.$$