Fall 2024 Math 541a Exam

Problem 1. For $n \geq 1$ let r_n be a sequence of known real numbers, not all zero, and

$$X_n = r_n\theta + \epsilon_n \quad \text{where} \quad \epsilon_n \sim \mathcal{N}(0,1) \tag{1}$$

are independent, and $\theta \in \mathbb{R}$ is unknown.

1. Find the maximum likelihood estimator $\widehat{\theta}_n$ of θ based on the observations $X_1, \ldots, X_n$, and determine its distribution.

2. Find a necessary and sufficient condition for $\widehat{\theta}_n$ to be a consistent estimator of θ, and prove they are such.

3. Now consider the same model (1) but where ϵ_n are independent and identically distributed with mean zero and variance 1. Determine if your condition in part 2 is still sufficient for consistency for the same $\widehat{\theta}_n$ (now the least squares estimator) and provide a proof of your claim.

Problem 2. Let $\phi(\cdot)$ be the probability density function (pdf) of the standard Gaussian distribution $N(0,1)$ on $\mathbb{R}$. Let $\mu \in \mathbb{R}$, $\sigma \in (0,\infty)$, and $\theta := (\mu, \sigma^2)^T$. Let $X_1, \ldots, X_n$ be a sample of i.i.d. random variables drawn from a mixture of Gaussian distributions with equal weights, whose pdf is given by

$$f(x;\theta) = \frac{1}{2}\phi(x-\mu) + \frac{1}{2\sigma}\phi\left(\frac{x-\mu}{\sigma}\right), \quad x \in \mathbb{R}.$$

That is, θ is the unknown parameter vector and $X_i \overset{\text{i.i.d.}}{\sim} \frac{1}{2}N(\mu,1) + \frac{1}{2}N(\mu,\sigma^2)$.

1. Find a method of moments estimator $\widehat{\theta} = (\widehat{\mu}, \widehat{\sigma^2})^T$ of θ.

2. Find the asymptotic distribution of the random vector $(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)^T$ after a suitable normalization. You may use the fact that: if $X \sim N(\mu,\sigma^2)$, then $\mathbb{E}X^3 = \mu^3 + 3\mu\sigma^2$ and $\mathbb{E}X^4 = 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4$.

3. Derive the asymptotic distribution of the vector $\widehat{\theta}$ (hint: you may find the multivariate Delta-method useful).