Topology qualifying exam Fall 2024

1. Let $\mathbb{RP}^2 \vee \mathbb{RP}^2$ denote the wedge sum of two copies of $\mathbb{RP}^2$. Explicitly, picking a basepoint $x_0 \in \mathbb{RP}^2$, we put

$$\mathbb{RP}^2 \vee \mathbb{RP}^2 := \left(\mathbb{RP}^2 \times \{x_0\}\right) \cup \left(\{x_0\} \times \mathbb{RP}^2\right) \subset \mathbb{RP}^2 \times \mathbb{RP}^2.$$

   (a) Compute the fundamental groups of $\mathbb{RP}^2 \vee \mathbb{RP}^2$ and $\mathbb{RP}^2 \times \mathbb{RP}^2$.

   (b) Prove that $\mathbb{RP}^2 \vee \mathbb{RP}^2$ is not a retract of $\mathbb{RP}^2 \times \mathbb{RP}^2$.

   (c) Prove that any map $\mathbb{RP}^2 \vee \mathbb{RP}^2 \to S^1$ is nullhomotopic.

   (d) Give an example of a map $\mathbb{RP}^2 \times \mathbb{RP}^2 \to \mathbb{RP}^2 \vee \mathbb{RP}^2$ which is not nullhomotopic.

2. Let $f : S^{2n-1} \to S^n$ be a smooth map, and let $\lambda \in \Omega^n(S^n)$ be an n-form with $\int_{S^n} \lambda = 1$.

   (a) Show that there exists an $\omega \in \Omega^{n-1}(S^{2n-1})$ such that $d\omega = f^*(\lambda)$.

   (b) Show that if $\omega' \in \Omega^{n-1}(S^{2n-1})$ is another form with $d\omega' = f^*(\lambda)$, then $\int_{S^{2n-1}} \omega \wedge f^*(\lambda) = \int_{S^{2n-1}} \omega' \wedge f^*(\lambda)$.

3. (a) Give an example of a path-connected topological space X whose fundamental group is nonzero and isomorphic to $H_1(X)$.

   (b) Let X be a connected CW complex and $A \subset X$ a subcomplex such that A is homotopy equivalent to S^3 and X/A is homotopy equivalent to S^5. Compute $H_n(X)$ for all n.

   (c) Give an example of a topological space X whose reduced homology groups are $\widetilde{H}_5(X) \cong \mathbb{Z}$, $\widetilde{H}_2(X) \cong \mathbb{Z}/2\mathbb{Z}$, and $\widetilde{H}_n(X) \cong 0$ for $n \neq 2, 5$.

4. Let G be a topological group. This means G is a group which is also equipped with a topology such that the multiplication map $G \times G \to G, (g, h) \mapsto g \cdot h$ and inversion map $G \to G, g \mapsto g^{-1}$ are both continuous. Assuming that G is connected, prove that the fundamental group of G is abelian.