Algebra Qualifying Exam - Fall 2024

1. Show that the Galois group of the polynomial $f(x) = x^5 - 21x^2 + 6$ over $\mathbb{Q}$ is isomorphic to S_5. (Hint: show that the Galois group contains a 5-cycle and a transposition.)

2. Assume R is a Noetherian commutative unital ring and M is a finitely generated R-module. Prove that there exist an integer n, an increasing sequence of submodules

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

   and prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ together with R-module isomorphisms $M_i/M_{i-1} \cong R/\mathfrak{p}_i$, $i \geq 1$.

3. Consider the subgroup of $GL_2(\mathbb{R})$ (i.e., the group of invertible 2×2-matrices with real number entries) consisting of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

   Show that this group is solvable. Is this group nilpotent? Prove or disprove.

4. Let R be an Artinian ring and $J \subseteq R$ be the Jacobson radical in R. Suppose that $J^2 = J$. Prove that R is isomorphic to a finite product of matrix rings $A \cong \prod_{i=1}^{m} M_{n_i}(D_i)$, where each D_i is a division ring.

5. Suppose R is a commutative ring. Recall that if $S \subset R$ is a multiplicatively closed subset, then the localization $R[S^{-1}]$ is the collection of all formal fractions $\frac{r}{s}$ modulo the equivalence relation $\frac{r}{s} = \frac{r'}{s'}$ if $\exists s'' \in S$ such that $s''(rs' - sr') = 0$. The function $r \mapsto \frac{r}{1}$ yields a well-defined function $\pi : R \to R[S^{-1}]$.

   (a) Show that π is an isomorphism if and only if $S \subset R^{\times}$ (the multiplicative group of units in R).

   (b) Suppose R is a finite ring. Show that π need not be an isomorphism, but it is always surjective.

6. (a) For an ideal $I \subseteq \mathbb{C}[x_1, x_2]$ write $V(I) \subset \mathbb{A}^2_{\mathbb{C}}$ for the variety of the ideal I, i.e., the collection of $\mathbf{x} \in \mathbb{C}^2$ such that $f(\mathbf{x}) = 0$ for all $f \in I$. Likewise, write $I(V)$ for the vanishing ideal of a subset $V \subset \mathbb{C}^2$, i.e., the collection of all $f \in \mathbb{C}[x_1, x_2]$ such that $f(\mathbf{x}) = 0$ for all $\mathbf{x} \in V$. Provide an explicit example of an ideal I for which $V(I)$ is the union of two distinct lines in $\mathbb{C}^2$ and for which $I \neq I(V(I))$ (by a line, we mean the vanishing locus of a linear function $a_1x_1 + a_2x_2 + a_3$, where at least one of a_1, a_2 is non-zero.)

   (b) Provide an example of a 4-dimensional commutative $\mathbb{Q}$-algebra A which admits no ring homomorphism $A \to \mathbb{Q}$.