MATH 505a **QUALIFYING EXAM** **August 2024**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

(1) Suppose that X is a positive integer random variable with distribution

$$\mathbb{P}(X = x) = \frac{1}{(e-1)x!} \text{ for } x = 1, 2, 3, \dots.$$

(a) Find the probability generating function $g_X(u) = \mathbb{E}[u^X]$.

(b) Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

(c) Let $U_1, U_2, \dots$ be a sequence of i.i.d. uniform random variables on $[0, 1]$ that is independent of X. Define $M = \max(U_1, \dots, U_X)$. Compute the c.d.f. of M.

(2) Do the following problems.

(a) Let $Z_1, Z_2, \dots$ be i.i.d. exponential random variables with parameter $\lambda > 0$, i.e. $\mathbb{P}(Z_1 > t) = e^{-\lambda t}$ for all $t > 0$. Put $S_n = Z_1 + \cdots + Z_n$. Find the joint density of $(S_1/S_{n+1}, \dots, S_n/S_{n+1})$.

Hint: Show first that when non-zero the joint density of $S_1, \dots, S_n, S_{n+1}$ depends only on s_{n+1}, and use this to figure out what is the conditional distribution of $S_1, \dots, S_n$, given that $S_{n+1} = s_{n+1} > 0$.

(b) Let $U_1, \dots, U_5$ be i.i.d. uniform $(0, 1)$ random variables. Let $U_{[1]}, \dots, U_{[5]}$ be $U_1, \dots, U_5$ sorted in ascending order. Find $\mathbb{P}\left(U_{[1]} + U_{[5]} \le 2U_{[3]}\right)$. Hint: Use part (a).

(3) There are $n \ge 6$ people in a room. The people are numbered from 1 to n, and each is wearing a hat with their number written on it. The people throw their hats into the middle of the room and then each person picks a different hat, with all permutations equally likely. Let N be the number of pairs (i, j) with $1 \le i < j \le n$ such that person i picked hat j and person j picked hat i. Let M be the number of triples (k, l, m) with $1 \le k < l < m \le n$ such that person k got hat l, person l got hat m and person m got hat k.

(a) Find $\mathbb{E}[N]$ and $\mathbb{E}[M]$.

(b) Find the covariance $\mathrm{Cov}(M, N)$ of M and N.

(c) Are M and N independent? Justify your answer.