Numerical Analysis Preliminary Examination Fall 2024

August 12, 2024

Problem 1.
Suppose $A \in \mathbb{R}^{n\times n}$ is invertible with non-zero elements along its diagonal and write $A = L+U$ where L is lower triangular and U is upper triangular with zeros along the diagonal. Recall the Gauss-Seidel method for finding the solution of $Ax = b$ is

$$Lx_k = b - Ux_{k-1}.$$

Let x^* be the exact solution to $Ax = b$.

(a) Let $e_k = x_k - x^*$ be the error in the k'th iterate. Show there is a matrix T such that

$$e_k = T^k e_0$$

and identify the matrix T in terms of L and U.

(b) Show the method converges for all initial vectors x_0 if and only if $\rho(T) < 1$ (where $\rho(T)$ is the spectral radius of T).

(c) Suppose there is exactly one eigenvalue λ of T for which $|\lambda| > 1$. Explain why the Gauss-Seidel method diverges for essentially all initial vectors, x_0.

(d) Suppose A is diagonally dominant by rows. Show the Gauss-Seidel method converges for all initial vectors x_0. *Hint: Write* $\det(T - \lambda I)$ *in terms of* $\det(U + \lambda L)$. *What can you conclude about* $\det(T - \lambda I)$ *when* $|\lambda| \geq 1$*?*

Problem 2.
Let A, T be linear maps from C^n into C^m and $B = A + T$. Denote the pseudo-inverse of A by $A^\dagger$, the adjoint of A by A^H, the column space of A by $C(A)$ and the nullspace of A by $N(A)$. We would like to look into the sensitivity of $A^\dagger$ when A is perturbed by T.

(a) Let α, β be nonzero numbers and let A, T be as follows

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}$$

Compute $\|T\|_2$, $rank(A)$, $rank(B)$, $B^\dagger - A^\dagger$ and show that $\|B^\dagger - A^\dagger\|_2$ can be made arbitrarily large by choosing ϵ appropriately.

(b) Show that if $rank(B) = rank(A)$ and $\|T\|_2 < \frac{1}{\|A^\dagger\|_2}$ then

$$\|(A+T)^\dagger\|_2 \leq \frac{\|A^\dagger\|_2}{1 - \|A^\dagger\|_2\|T\|_2}$$

Hint: Use the fact that if $rank(A) = rank(B) = r$ and the singular values of A are $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_r(A) > 0$ then $\sigma_r(A+T) \geq \sigma_r(A) - \sigma_1(T)$. You may use this fact without proving it.

(c) In the case that $rank(B) = rank(A)$ we can also show that:

$$\|B^\dagger - A^\dagger\|_2 \leq \sqrt{2}\|B^\dagger\|_2\|A^\dagger\|_2\|B - A\|_2$$

Show the following decomposition which is central in the proof of this result

$$B^\dagger - A^\dagger = P_{C(B^H)}(B^\dagger - A^\dagger)P_{C(A)} + B^\dagger P_{N(A^H)} - P_{N(B)}A^\dagger$$

Where the orthogonal projection onto a subspace X is denoted by P_X.

Problem 3.

Let $A, Q \in \mathbb{R}^{n\times n}$, with Q positive semi-definite symmetric, let $R \in \mathbb{R}^{m\times m}$ with R positive definite symmetric, and let $B \in \mathbb{R}^{n\times m}$. Set $Z = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$.

(a) Show that the characteristic polynomial of Z is in fact a polynomial in s^2 and therefore if $\lambda \in \sigma(Z)$ then $-\lambda \in \sigma(Z)$. (Hint: Use the elementary properties of the determinant to show that $det(-sI - Z) = det(sI - Z)$).

(b) Let $Z = \hat{W}\hat{J}\hat{W}^{-1}$ denote a complex Jordan decomposition of Z and partition $\hat{W}$ and $\hat{J}$ into $n\times n$ blocks with $\hat{J} = \begin{pmatrix} \hat{J}_{11} & \hat{J}_{12} \\ 0 & \hat{J}_{22} \end{pmatrix}$ and $\hat{W} = \begin{pmatrix} \hat{W}_{11} & \hat{W}_{12} \\ \hat{W}_{21} & \hat{W}_{22} \end{pmatrix}$. Find a $2n\times 2n$ block matrix with blocks of size $n\times n$, $W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$, such that $Z = WJW^{-1}$ with $J = \begin{pmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} \hat{J}_{22} & 0 \\ \hat{J}_{12} & \hat{J}_{11} \end{pmatrix}$.

(c) If W_{12} is invertible, let $P = W_{22}W_{12}^{-1}$ and show that P is a solution to the matrix algebraic Riccati equation (ARE) $PA + A^TP - PBR^{-1}B^TP + Q = 0$, and show that if the ARE has a unique solution, then P must be symmetric (Hint: Recall that $ZW = WJ$).

(d) With P as in part (c), show that the matrix $A - BR^{-1}B^TP$ is similar to J_{22} and that the (generalized) eigenvectors of $A - BR^{-1}B^TP$ are the columns of the matrix W_{12}.

(e) If Z is in fact diagonalizable (this is really not necessary) with no eigenvalues on the imaginary axis, show that there is a choice for the Jordan decomposition of Z (i.e. of $\hat{W}$ and $\hat{J}$) so that the matrix $A - BR^{-1}B^TP$ is a Hurwitz or stability matrix (a matrix having all its eigenvalues in the left half plane).