# PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM
## Spring 2023

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Consider the equation
$$u_x^2(x,y) + 2u_y^2(x,y) = x^2 + 2y^2,$$
   (a) Find at least one classical solution to this equation in $\mathbb{R}^2$ such that $u(x,x) = x^2$.
   (b) Is the problem from (a) uniquely solvable?

2. Let $u$ be harmonic in the domain $U = B(0,4)$ in $\mathbb{R}^2$.
   (a) Show that if $u$ is bounded then
$$\sup_{x \in U} (4 - |x|)|\nabla u(x)| < \infty;$$
   (b) Give an example of $u$ that is harmonic in $U$ and unbounded, but still satisfies the claim from (a).

3. Suppose $u$ is a $C^2$ function satisfying
$$\begin{cases} u_{tt} = \Delta u \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = g(x), \\ \partial_t u(x,0) = h(x). \end{cases}$$
Fix $T > 0$ and consider the set
$$K_T := \{(x,t) : 0 \le t \le T, |x| \le T - t\}.$$
Prove that, if $g(x) = h(x) = 0$ for all $x \in B(0,T)$, then $u(x,t) = 0$ for all $(x,t) \in K_T$.

# PDE Screening exam Spring 2024

December 10, 2023

**Problem 1.** Find a classical solution $u(x, y)$ of the equation

$$(\partial_x u)^2 - x^2 = (\partial_y u)^2 - y^2$$

in $\mathbb{R}^2$ satisfying the boundary condition $u(y, y) = y^2$.

**Problem 2.** Let $U$ be open and bounded in $\mathbb{R}^n$. Suppose $u$ and $\Delta u$ are continuous in $U$, and $u$ satisfies that

$$\begin{cases} \Delta u = u^4 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

(1) If $u \geq 0$ in $\overline{U}$, prove that $u \equiv 0$.

(2) If the condition $u \geq 0$ in $\overline{U}$ is absent, what can you say about $u(x)$?

**Problem 3.** Let $P, Q, R, S \in \mathbb{R} \times \mathbb{R}^+$ be consecutive vertices of a rectangle with sides parallel to the lines $x + t = 0$ and $x - t = 0$, respectively.

(1) Let $u = u(x, t) \in C^2(\mathbb{R} \times \mathbb{R}^+)$ be a solution to the wave equation in $\mathbb{R} \times \mathbb{R}^+$, show that

$$u(P) + u(R) = u(Q) + u(S).$$

(2) Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be given functions. Find a solution to the wave equation in $\mathbb{R}^2$ such that

$$u(t - 1, t) = \alpha(t), \quad u(5 - t, t) = \beta(t).$$

**PDE Screening Exam**
**Fall 2024**

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Consider a linear transport equation ($a > 0$) with drag ($b > 0$):

$$u_t + au_x + bu = 0. \tag{1}$$

   (a) Find the solution $u(t,x)$ to Eq. (1) for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ with the initial condition $u(0,x) = f(x)$.

   (b) Find the solution $u(t,x)$ to Eq. (1) for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+$, subject to the initial condition $u(0,x) = f(x)$ and the boundary condition $u(t,0) = g(t)$, where $f(0) = g(0)$.

2. Let $\Omega \subset \mathbb{R}^n$ $(n > 1)$ be a bounded domain with smooth boundary. Assume that $u \in C^2(\bar{\Omega})$ solves

$$\Delta u = u^7 + 2u^5 + 3u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

   Show that $u$ is identically zero.

3. Let $\Omega \subset \mathbb{R}^n$ $(n > 1)$ be a bounded domain with smooth boundary. Let $u \in C^2([0,\infty) \times \bar{\Omega})$, which solves the equation

$$u_{tt} - \Delta u = u$$

   with the boundary condition $u = 0$ on $\partial\Omega$. Let

$$E(t) = \frac{1}{2} \int_\Omega u_t^2(t,x) + |\nabla_x u(t,x)|^2 \, dx$$

   Prove that there exists $C > 0$ independent of $t$ such that

$$E(t) \le \exp(Ct)E(0) \quad \text{for } t \ge 0.$$

# PDE Screening Exam
# Spring 2025

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Let $u \not\equiv 0$ be a $C^2(\mathbb{R}^n)$ function ($n > 1$). Define

$$m_x(r) = r^{1-n} \int_{\partial B(x,r)} u(y)dS(y).$$

   (a) Show that

$$\frac{dm_x(r)}{dr} = r^{1-n} \int_{B(x,r)} \Delta u(y)dy.$$

   (b) Let $u$ solve $-\Delta u = \phi(u)$ for some continuous function $\phi$. Assume that $u(x) \geq 1$ for all $x \in \mathbb{R}^n$, and that $\phi(\xi) \geq 0$ for all $\xi \geq 1$. Using the result of part (a) above, prove that if $u(x_0) = 1$ for some $x_0 \in \mathbb{R}^n$, then $u(x) \equiv 1$ for all $x \in \mathbb{R}^n$.

2. Use the method of characteristics to solve the following partial differential equation:

$$\begin{aligned} \partial_t u - u\partial_x u &= 3u, \quad x \in \mathbb{R}, \quad t > 0, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}. \end{aligned}$$

3. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded domain with smooth boundary. Assume that $u(t,x) \geq 0$ is a smooth function solving

$$\begin{cases} \partial_t u - \Delta u = -u^4 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

with $u|_{t=0} = u_0$. Here $\nu$ is the unit normal to the boundary. Let

$$E(t) = \int_\Omega u^2(t,x)dx.$$

Show that there exists a constant $C > 0$ such that for each $t > 0$

$$E(t) \leq \frac{1}{(E^{-\frac{3}{2}}(0) + Ct)^{\frac{2}{3}}}$$