

Geometry/Topology Qualifying Exam

Spring 2012

Solve all **SEVEN** problems. Partial credit will be given to partial solutions.

- (10 pts) Prove that a compact smooth manifold of dimension n cannot be immersed in \mathbb{R}^n .
- (10 pts) Let $\Sigma_{1,1}$ be the compact oriented surface with boundary, obtained from $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates (x, y) by removing a small disk $\{(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{100}\}$.
 - Compute the homology of $\Sigma_{1,1}$.
 - Let Σ_2 denote a closed oriented surface of genus 2. Use your answer from (a) to compute the homology of Σ_2 .
- (10 pts) Let S be an oriented embedded surface in \mathbb{R}^3 and ω be an area form on S which satisfies $\omega(p)(e_1, e_2) = 1$ for all $p \in S$ and any orthonormal basis (e_1, e_2) of $T_p S$ with respect to the standard Euclidean metric on \mathbb{R}^3 . If (n_1, n_2, n_3) is the unit normal vector field of S , then prove that

$$\omega = n_1 dy \wedge dz - n_2 dx \wedge dz + n_3 dx \wedge dy,$$

where (x, y, z) are the standard Euclidean coordinates on \mathbb{R}^3 .

- (10 pts) Consider the space $X = M_1 \cup M_2$, where M_1 and M_2 are Möbius bands and $M_1 \cap M_2 = \partial M_1 = \partial M_2$. Here a *Möbius band* is the quotient space $([-1, 1] \times [-1, 1]) / ((1, y) \sim (-1, -y))$.
 - Determine the fundamental group of X .
 - Is X homotopy equivalent to a compact orientable surface of genus g for some g ?
- (10 pts) Determine all the connected covering spaces of $\mathbb{R}P^{14} \vee \mathbb{R}P^{15}$.
- (10 pts) Let $f : M \rightarrow N$ be a smooth map between smooth manifolds, X and Y be smooth vector fields on M and N , respectively, and suppose that $f_* X = Y$ (i.e., $f_*(X(x)) = Y(f(x))$ for all $x \in M$). Then prove that $f^*(\mathcal{L}_Y \omega) = \mathcal{L}_X(f^* \omega)$, where ω is a 1-form on N . Here \mathcal{L} denotes the Lie derivative.
- (10 pts) Consider the linearly independent vector fields on $\mathbb{R}^4 - \{0\}$ given by:

$$X(x_1, x_2, x_3, x_4) = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}$$
$$Y(x_1, x_2, x_3, x_4) = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}.$$

Is the rank 2 distribution orthogonal to these two vector fields integrable? Here orthogonality is measured with respect to the standard Euclidean metric on \mathbb{R}^4 .

Geometry-Topology Qualifying exam Fall 2012

Solve all of the problems. Partial credit will be given for partial answers.

- Denote by $S^1 \subset \mathbf{R}^2$ the unit circle and consider the torus $T^2 = S^1 \times S^1$. Now, define $A \subset T^2 = S^1 \times S^1$ by

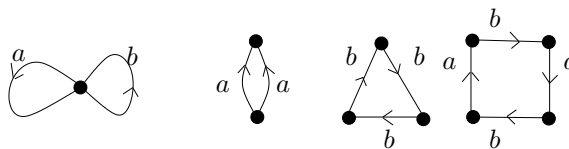
$$A = \{(x, y, z, w) \in T^2 \mid (x, y) = (0, 1) \text{ or } (z, w) = (0, 1)\}.$$

Compute $H^*(T^2, A)$. Here we regard S^1 as a subset of the plane, hence we indicate points on S^1 as ordered pairs.

- Denote by S^1 and S^2 the circle and sphere respectively. Recall that the definition of the smash product $X \wedge Y$ of two pointed spaces is the quotient of $X \times Y$ by $(x, y_0) \sim (x_0, y)$.

Show that $S^1 \times S^1$ and $S^1 \wedge S^1 \wedge S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

- Let X be a CW-complex with one vertex, two one cells and 3 two cells whose attaching maps are indicated below.



1-skeleton

2-skeleton

- Compute the homology of X .
- Present the fundamental group of X and prove its nonabelian.

(Justify your work.)

- Does there exist a smooth embedding of the projective plane $\mathbf{R}P^2$ into \mathbf{R}^2 ? Justify your answer.
- Let M be a manifold, and let $C^\infty(M)$ be the algebra of C^∞ functions $M \rightarrow \mathbf{R}$. Explain the relationship between vector fields on M and $C^\infty(M)$. If we consider the vector fields X and Y as maps $C^\infty(M) \rightarrow C^\infty(M)$ is the composition map XY also a vector field? What about $[X, Y] = XY - YX$? Explain.
- Let S be the unit sphere defined by $x^2 + y^2 + z^2 + w^2 = 1$ in \mathbf{R}^4 . Compute $\int_S \omega$ where $\omega = (w + w^2)dx \wedge dy \wedge dz$.
- Does the equation $x^2 = y^3$ define a smooth submanifold in \mathbf{R}^3 ? Prove your claim.

GEOMETRY TOPOLOGY QUALIFYING EXAM SPRING 2013

Solve all of the problems that you can. Partial credit will be given for partial solutions.

- (1) Consider the form

$$\omega = (x^2 + x + y)dy \wedge dz$$

on \mathbb{R}^3 . Let $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be the unit sphere, and $i: S^2 \rightarrow \mathbb{R}^3$ the inclusion.

- (a) Calculate $\int_{S^2} \omega$.
- (b) Construct a closed form α on \mathbb{R}^3 such that $i^* \alpha = i^* \omega$, or show that such a form α does not exist.
- (2) Find all points in \mathbb{R}^3 in a neighborhood in which the functions $x, x^2 + y^2 + z^2 - 1, z$ can serve as a local coordinate system.
- (3) Prove that the real projective space $\mathbb{R}P^n$ is a smooth manifold of dimension n .
- (4) (a) Show that every closed 1-form on S^n , $n > 1$ is exact.
(b) Use this to show that every closed 1-form on $\mathbb{R}P^n$, $n > 1$ is exact.
- (5) Let X be the space obtained from \mathbb{R}^3 by removing the three coordinate axes. Calculate $\pi_1(X)$ and $H_*(X)$.
- (6) Let $X = T^2 - \{p, q\}$, $p \neq q$ be the twice punctured 2-dimensional torus.
(a) Compute the homology groups $H_*(X, \mathbb{Z})$.
(b) Compute the fundamental group of X .
- (7) (a) Find all of the 2-sheeted covering spaces of $S^1 \times S^1$.
(b) Show that if a path-connected, locally path connected space X has $\pi_1(X)$ finite, then every map $X \rightarrow S^1$ is nullhomotopic.
- (8) (a) Show that if $f: S^n \rightarrow S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$.
(b) Show that if X has S^{2n} as universal covering space then $\pi_1(X) = \{1\}$ or \mathbb{Z}_2 .

Geometry/Topology Qualifying Exam

Fall 2013

Solve all **SEVEN** problems. Partial credit will be given to partial solutions.

- (15 pts) Let X denote S^2 with the north and south poles identified.
 - (5 pts) Describe a cell decomposition of X and use it to compute $H_i(X)$ for all $i \geq 0$.
 - (5 pts) Compute $\pi_1(X)$.
 - (5 pts) Describe (i.e., draw a picture of) the universal cover of X and all other connected covering spaces of X .
- (10 pts) Show that if M is compact and N is connected, then every submersion $f : M \rightarrow N$ is surjective.
- (10 pts) Show that the orthogonal group $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = id\}$ is a smooth manifold. Here $M_n(\mathbb{R})$ is the set of $n \times n$ real matrices.
- (10 pts) Compute the de Rham cohomology of $S^1 = \mathbb{R}/\mathbb{Z}$ from the definition.
- (10 pts) Let X, Y be topological spaces and $f, g : X \rightarrow Y$ two continuous maps. Consider the space Z obtained from the disjoint union $(X \times [0, 1]) \sqcup Y$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form:
$$\dots \rightarrow H_n(X) \rightarrow H_n(Y) \rightarrow H_n(Z) \rightarrow H_{n-1}(X) \rightarrow \dots$$
- (10 pts) A lens space $L(p, q)$ is the quotient of $S^3 \subset \mathbb{C}^2$ by the $\mathbb{Z}/p\mathbb{Z}$ -action generated by $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$ for coprime p, q .
 - (5 pts) Compute $\pi_1(L(p, q))$.
 - (5 pts) Show that any continuous map $L(p, q) \rightarrow T^2$ is null-homotopic.
- (10 pts) Consider the space of all straight lines in \mathbb{R}^2 (not necessarily those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?

Geometry and Topology Graduate Exam
Spring 2014

Solve all SEVEN problems. Partial credit will be given to partial solutions.

Problem 1. Let X_n denote the complement of n distinct points in the plane \mathbb{R}^2 . Does there exist a covering map $X_2 \rightarrow X_1$? Explain.

Problem 2. Let $D = \{z \in \mathbb{C}; |z| \leq 1\}$ denote the unit disk, and choose a base point z_0 in the boundary $S^1 = \partial D = \{z \in \mathbb{C}; |z| = 1\}$. Let X be the space obtained from the union of D and $S^1 \times S^1$ by gluing each $z \in S^1 \subset D$ to the point $(z, z_0) \in S^1 \times S^1$. Compute all homology groups $H_k(X; \mathbb{Z})$.

Problem 3. Let $B^n = \{x \in \mathbb{R}^n; \|x\| \leq 1\}$ denote the n -dimensional closed unit ball, with boundary $S^{n-1} = \{x \in \mathbb{R}^n; \|x\| = 1\}$. Let $f: B^n \rightarrow \mathbb{R}^n$ be a continuous map such that $f(x) = x$ for every $x \in S^{n-1}$. Show that the origin 0 is contained in the image $f(B^n)$. (Hint: otherwise, consider $S^{n-1} \rightarrow B^n \xrightarrow{f} \mathbb{R}^n - \{0\}$.)

Problem 4. Consider the following vector fields defined in \mathbb{R}^2 :

$$\mathbf{X} = 2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad \text{and} \quad \mathbf{Y} = \frac{\partial}{\partial y}.$$

Determine whether or not there exists a (locally defined) coordinate system (s, t) in a neighborhood of $(x, y) = (0, 1)$ such that

$$\mathbf{X} = \frac{\partial}{\partial s}, \quad \text{and} \quad \mathbf{Y} = \frac{\partial}{\partial t}.$$

Problem 5. Let M be a differentiable (not necessarily orientable) manifold. Show that its cotangent bundle

$$T^*M = \{(x, u); x \in M \text{ and } u: T_x M \rightarrow \mathbb{R} \text{ linear}\}$$

is a manifold, and is orientable.

Problem 6. Calculate the integral $\int_{S^2} \omega$ where S^2 is the standard unit sphere in \mathbb{R}^3 and where ω is the restriction of the differential 2-form

$$(x^2 + y^2 + z^2)(x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy)$$

Problem 7. Let M be a compact m -dimensional submanifold of $\mathbb{R}^m \times \mathbb{R}^n$. Show that the space of points $x \in \mathbb{R}^m$ such that $M \cap \mathbb{R}^n$ is infinite has measure 0 in \mathbb{R}^m .

Geometry/Topology Qualifying Exam - Fall 2014

1. Show that if (X, x) is a pointed topological space whose universal cover exists and is compact, then the fundamental group $\pi_1(X, x)$ is a finite group.
2. Recall that if (X, x) and (Y, y) are pointed topological spaces, then the wedge sum (or 1-point union) $X \vee Y$ is the space obtained from the disjoint union of X and Y by identifying x and y . Show that T^2 (the 2-torus $S^1 \times S^1$) and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups, but are not homeomorphic.
3. Suppose S^n is the standard unit sphere in Euclidean space and that $f : S^n \rightarrow S^n$ is a continuous map.
 - i) Show that if f has no fixed points, then f is homotopic to the antipodal map.
 - ii) Show that if $n = 2m$, then there exists a point $x \in S^{2m}$ such that either $f(x) = x$ or $f(x) = -x$.
4. If M is a smooth manifold of dimension d , using basic properties of de Rham cohomology, describe the de Rham cohomology groups $H_{dR}^*(S^1 \times M)$ in terms of the groups $H_{dR}^*(M)$ (along the way, please explain, quickly and briefly, how to compute $H_{dR}^*(S^1)$).
5. Show that if $X \subset \mathbb{R}^3$ is a closed (i.e., compact and without boundary) submanifold that is homeomorphic to a sphere with $g > 1$ handles attached, then there is a non-empty open subset on which the Gaussian curvature K is negative.
6. Suppose M is a (non-empty) closed oriented manifold of dimension d . Show that if ω is a differential d -form, and X is a (smooth) vector field on X , then the differential form $\mathcal{L}_X \omega$ necessarily vanishes at some point of M .
7. Let V be a 2-dimensional complex vector space, and write $\mathbb{C}\mathbb{P}^1$ for the set of complex 1-dimensional subspaces of V . By explicit construction of an atlas, show that $\mathbb{C}\mathbb{P}^1$ can be equipped with the structure of an oriented manifold.

Geometry and Topology Graduate Exam
Fall 2015

Problem 1. (15 points)

- (a) Define the two notions “homotopy between two maps” and “homotopy equivalences between two spaces”.
- (b) Give an example of two topological spaces X and Y that are homotopy equivalent but are not homeomorphic.
- (c) Give an example of path-connected topological spaces X and Y that have isomorphic fundamental groups but are not homotopy equivalent.
- (d) Give an example of path-connected topological spaces X and Y that have isomorphic first homology groups $H_1(X; \mathbb{Z}) \cong H_1(Y; \mathbb{Z})$ but whose fundamental groups are not isomorphic.

Problem 2. (15 points) Let T be the 2-dimensional torus, and let K be the Klein bottle.

- (a) Describe a twofold covering map $p: T \rightarrow K$. (“Twofold” means that the preimage of each point of K consists of two points of T .)
- (b) Pick base points $x_0 \in T$ and $y_0 \in K$ such that $y_0 = p(x_0)$. Give generators for the fundamental groups $\pi_1(T; x_0)$ and $\pi_1(K; y_0)$ and, for each generator of $\pi_1(T; x_0)$, express its image under the induced homomorphism $p_*: \pi_1(T; x_0) \rightarrow \pi_1(K; y_0)$ in terms of the generators of $\pi_1(K; y_0)$.

Problem 3. (25 points) Let Σ_g and $\Sigma_{g'}$ be closed orientable surfaces of genus g and $g' > 0$, respectively. Let $f: B^2 \rightarrow \Sigma_g$ be an embedding of the 2-dimensional disk B^2 , and consider the simple closed curve $\gamma = f(S^1) \subset \Sigma_g$. Similarly, let $\gamma' = f'(S^1) \subset \Sigma_{g'}$ be associated to an embedding $f': B^2 \rightarrow \Sigma_{g'}$. Finally, let X be the topological space obtained by gluing Σ_g and $\Sigma_{g'}$ along γ and γ' ; namely, X is obtained from the disjoint union $\Sigma_g \sqcup \Sigma_{g'}$ by gluing $f(x)$ to $f'(x)$ for every $x \in S^1$.

- (a) Compute the fundamental group of X .
- (b) Compute all homology groups of X .
- (c) Is X homotopy equivalent to the product $\Sigma_g \times \Sigma_{g'}$?

Problem 4. (15 points) Let M be a manifold of dimension n , and let ω be a differential form of degree $n - 1$ on M . Suppose that $\int_N \omega = 0$ for every $(n - 1)$ -dimensional oriented closed submanifold N of M . Show that $d\omega = 0$. (Possible hint: look at small spheres.)

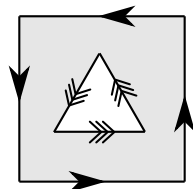
Problem 5. (15 points) Consider the vector fields $\mathbf{v} = \partial_x + xz\partial_z$ and $\mathbf{w} = \partial_y + yz\partial_z$ in \mathbb{R}^3 . If P is a point of \mathbb{R}^3 , does there exist a local coordinate system in a neighborhood of P in which \mathbf{v} and \mathbf{w} ? Namely, is there a diffeomorphism $\phi: U \rightarrow V$ from a neighborhood U of P to an open subset $V \subset \mathbb{R}^3$ that sends \mathbf{v} to ∂_x and \mathbf{w} to ∂_y ?

Problem 6. (15 points) Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial of one complex variable. Recall that the one-point compactification $\mathbb{C} \cup \{\infty\}$ of \mathbb{C} is homeomorphic to the sphere S^2 .

- (a) Show that f extends to a continuous map $\bar{f}: S^2 \rightarrow S^2$.
- (b) Show that the degree of \bar{f} (in the sense of topology or geometry) is equal to the degree of the polynomial f (in the algebraic sense).

Geometry and Topology Graduate Exam
Spring 2016

Problem 1. Let Y be the space obtained by removing an open triangle from the interior of a compact square in \mathbb{R}^2 . Let X be the quotient space of Y by the equivalence relation which identifies all four edges of the square and which identifies all three edges of the triangle according to the diagram below. Compute the fundamental group of X .



Problem 2. Let X be a path connected space with $\pi_1(X; x_0) = \mathbb{Z}/5$, and consider a covering space $\pi : \tilde{X} \rightarrow X$ such that $p^{-1}(x_0)$ consists of 6 points. Show that \tilde{X} has either 2 or 6 connected components.

Problem 3. Compute the homology groups $H_k(S^1 \times S^n; \mathbb{Z})$ of the product of the circle S^1 and the sphere S^n , with $n \geq 1$.

Problem 4. Let M be a compact oriented manifold of dimension n , and consider a differentiable map $f : M \rightarrow \mathbb{R}^n$ whose image $f(M)$ has non-empty interior in \mathbb{R}^n .

- (a) Show there is at least one point $x \in M$ where f is a local diffeomorphism, namely such that there exists an open neighborhood $U \subset M$ of x such that restriction $f|_U : U \rightarrow f(U)$ is a diffeomorphism.
- (b) Show that there exists at least two points $x, y \in M$ such that f is a local diffeomorphism at x and y , f is orientation-preserving at x , and f is orientation-reversing at y . Possible hint: What is the degree of f ?

Problem 5. Consider the real projective space $\mathbb{R}P^n$, quotient of the sphere S^n by the equivalence relation that identifies each $x \in S^n$ to $-x$. Is there a degree n differential form such $\omega \in \Omega^n(\mathbb{R}P^n)$ such that $\omega(y) \neq 0$ at every $y \in \mathbb{R}P^n$? (The answer may depend on n .)

Problem 6. Let S^n denote the n -dimensional sphere, and remember that for $n \geq 1$ its de Rham cohomology groups are

$$H^k(S^n) \cong \begin{cases} 0 & \text{if } k \neq 0, n \\ \mathbb{R} & \text{if } k = 0, n. \end{cases}$$

Consider a differentiable map $f : S^{2n-1} \rightarrow S^n$, with $n \geq 2$. If $\alpha \in \Omega^n(S^n)$ is a differential form of degree n on S^n such that $\int_{S^n} \alpha = 1$, let $f^*(\alpha) \in \Omega^n(S^{2n-1})$ be its pull-back under the map f .

- (a) Show that there exists $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.
- (b) Show that the integral $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choice of β and α .

Geometry and Topology Graduate Exam
Spring 2017

Problem 1. Let $\omega \in \Omega^2(M)$ be a differential form of degree 2 on a $2n$ -dimensional manifold M . Suppose that ω is exact, namely that $\omega = d\alpha$ for some $\alpha \in \Omega^1(M)$. Show that $\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M)$ is exact.

Problem 2. Consider the unit disk $B^2 = \{x \in \mathbb{R}^2; \|x\| \leq 1\}$ and the circle $S^1 = \{x \in \mathbb{R}^2; \|x\| = 1\}$. The two manifolds $U = S^1 \times B^2$ and $V = B^2 \times S^1$ have the same boundary $\partial U = \partial V = S^1 \times S^1$. Let X be the space obtained by gluing U and V along this common boundary; namely, X is the quotient of the disjoint union $U \sqcup V$ under the equivalence relation that identifies each point of ∂U to the point of ∂V that corresponds to the same point of $S^1 \times S^1$.

Compute the fundamental group $\pi_1(X; x_0)$.

Problem 3. Compute the homology groups $H_n(X; \mathbb{Z})$ of the topological space X of Problem 2.

Problem 4. For a unit vector $v \in S^{n-1} \subset \mathbb{R}^n$, let $\pi_v: \mathbb{R}^n \rightarrow v^\perp$ be the orthogonal projection to its orthogonal hyperplane $v^\perp \subset \mathbb{R}^n$. Let M be an m -dimensional submanifold of \mathbb{R}^n with $m \leq \frac{n}{2} - 1$. Show that, for almost every $v \in S^{n-1}$, the restriction of π_v to M is injective.

Possible hint: Use a suitable map $f: M \times M - \Delta \rightarrow S^{n-1}$, where $\Delta = \{(x, x); x \in M\}$ is the diagonal of $M \times M$.

Problem 5. Let G be a topological group. Namely, G is simultaneously a group and a topological space, the multiplication map $G \times G \rightarrow G$ defined by $(g, h) \mapsto gh$ is continuous, and the inverse map $G \rightarrow G$ defined by $g \mapsto g^{-1}$ is continuous as well. Show that, if $e \in G$ is the identity element of G , the fundamental group $\pi_1(G; e)$ is abelian.

Problem 6. Let $f: M \rightarrow N$ be a differentiable map between two compact connected oriented manifolds M and N of the same dimension m . Show that, if the induced homomorphism $H_m(f): H_m(M; \mathbb{Z}) \rightarrow H_m(N; \mathbb{Z})$ is nonzero, the subgroup $f_*(\pi_1(M; x_0))$ has finite index in $\pi_1(N; f(x_0))$. Hint: consider a suitable covering of N .

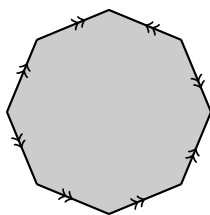
Geometry and Topology Graduate Exam
Fall 2017

Solve all SEVEN problems. Partial credit will be given to partial solutions.

Problem 1. Let M be an oriented compact m -dimensional manifold, and let $f: M \rightarrow \mathbb{R}^m$ be a smooth map. Show that, for almost every $y \in \mathbb{R}^m$ (meaning, for y in the complement of a set of measure 0), the preimage $f^{-1}(y)$ consists of an *even* number of points.

Problem 2.

The sides of an octagon are glued using the pattern below. Determine the fundamental group of the associated quotient space.



Problem 3. Let $p: \tilde{X} \rightarrow X$ be a covering map with X path connected and locally path connected, and with $\pi_1(X; x_0) \cong \mathbb{Z}/5$. Show that, if the fiber $p^{-1}(x_0)$ consists of 4 points, the covering is trivial.

Problem 4. Consider the following two-dimensional distribution on \mathbb{R}^3 :

$$\mathcal{D} = \ker(2dx - e^y dz).$$

Is there a neighborhood U of $0 \in \mathbb{R}^3$, along with a coordinate system (w, s, t) on U , such that $\mathcal{D}|_U = \text{span}(\frac{\partial}{\partial w}, \frac{\partial}{\partial s})$? Justify your answer (with a proof).

Problem 5. Show that the subset

$$M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_1^2 + x_2^2 = x_3^2 + x_4^2\}$$

is *not* a differentiable submanifold of \mathbb{R}^4 .

Problem 6. Let X be the subspace of \mathbb{R}^3 defined by

$$X = \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2 - 1)(x^2 + z^2 - (\frac{1}{2})^2) = 0, \}$$

so that X is the union of two cylinders of radius 1 along the z -axis and a cylinder of radius $\frac{1}{2}$ along the y -axis. Determine the homology groups $H_*(X)$.

Problem 7. Consider the 2-form on $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ given by

$$\sigma = \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} (x dy \wedge dz - y dx \wedge dz + z dx \wedge dy).$$

Show that σ is closed but not exact. (Possible hint: Integrate σ over the sphere S^2).

Geometry and Topology Graduate Exam
Spring 2018

Solve all 7 problems. Partial credit will be given to partial solutions.

Problem 1.

a) Define $F: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ by

$$F(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 + x_3^2 + x_4^2, x_1^2 + x_2^2 - x_3^2 - x_4^2).$$

Show that $M = F^{-1}(1, 0)$ is a smooth manifold.

b) For each $x = (x_1, x_2, x_3, x_4) \in M$ show that the tangent space $T_x M$ is spanned by $(x_2, -x_1, 0, 0)$ and $(0, 0, x_4, -x_3)$.

c) Let $G: \mathbb{R}^4 \rightarrow \mathbb{R}$ be a smooth map and $g = G|_M$ be the restriction of G to M . Show that x is a critical point of g if and only if $\ker dF_x \subset \ker dG_x$.

d) If $G(x_1, x_2, x_3, x_4) = x_1 + x_3$ find the critical points of g .

Problem 2. Let X be a topological space. Let SX denote the suspension of X , i.e. the space obtained from $X \times [0, 1]$ by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another point:

$$SX := X \times [0, 1] / \sim, \quad \text{where } \{(x, t) \sim (y, s) \text{ if } s = t = 0 \text{ or } s = t = 1 \text{ or } (x, t) = (y, s)\}$$

Determine the relationship between the homology of SX and X .

Problem 3. Consider \mathbb{R}^3 with the coordinates (x, y, z) . Write down explicit formulas for the vector fields X and Y which represent the infinitesimal generators of rotation about the x and y axes respectively and compute their Lie bracket.

Problem 4.

Draw a based covering space of the figure eight with each of the following subgroups of $\mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$ as its fundamental group. In each case determine whether or not the subgroup is normal.

- (1) $\langle a^3, b, aba^{-1}, a^{-1}ba \rangle$
- (2) $\langle a^2, b^2, aba, bab \rangle$

Problem 5.

a) Prove there does not exist a degree 1 map $S^2 \rightarrow T^2$.

b) Let $f: M^p \rightarrow N^p$ be any smooth map between two connected compact orientable manifolds of the same dimension p . Suppose there exists a regular value $y \in N$ with three pre-images $f^{-1}(y) = \{x_1, x_2, x_3\}$. Prove that f is necessarily surjective.

Problem 6.

Let $T = S^1 \times S^1$ and let $x, y \in T$ be two distinct points. Let Y be the quotient space obtained from $T \times \{1, 2\}$ by identifying the points $(x, 1)$ and $(x, 2)$ into a single point \bar{x} , and identifying the points $(y, 1)$ and $(y, 2)$ into a single point \bar{y} , distinct from \bar{x} . Compute $\pi_1(Y, \bar{x})$.

ADDITIONAL PROBLEM ON NEXT PAGE

Problem 7.

a) Compute the integral

$$\int_{S^2} 2xyz dx \wedge dy + (yz + xy^2) dx \wedge dz + xz dy \wedge dz,$$

where S^2 is oriented as the boundary of the unit ball $B^3 \subset \mathbb{R}^3$.

b) Show that the form

$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$

on $\mathbb{R}^2 - \{(0, 0)\}$ is closed but not exact.

Geometry and Topology Graduate Exam
Fall 2018

Solve all 6 problems. Partial credit will be given to partial solutions.

Problem 1. Let $p : \tilde{X} \rightarrow X$ denote the universal cover of the space X , and let $Y \subset X$ be a path-connected subspace. Suppose that the preimage $p^{-1}(Y)$ is path connected. Show that, for an arbitrary base point $y_0 \in Y$, the homomorphism $i_* : \pi_1(Y; y_0) \rightarrow \pi_1(X; y_0)$ induced by the inclusion map $i : Y \rightarrow X$ is surjective.

Problem 2. Consider the Klein bottle

$$K = S^1 \times [0, 1] / \sim$$

where, if S^1 is the unit circle in the complex plane \mathbb{C} and if \bar{z} denotes the complex conjugate of $z \in S^1$, the equivalence relation \sim identifies each $(z, 1) \in S^1 \times \{1\}$ to $(\bar{z}, 0) \in S^1 \times \{0\}$. Compute all homology groups $H_n(K; \mathbb{Z}_4)$ with coefficients in the cyclic group \mathbb{Z}_4 of order 4.

Problem 3. Consider the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

where I_n denotes the identity matrix of order n , and where 0_n represents the square matrix of order n whose entries are all equal to 0. In the vector space $M_{2n}(\mathbb{R}) \cong \mathbb{R}^{4n^2}$ of $2n \times 2n$ matrices, set

$$\Sigma_n = \{A \in M_{2n}(\mathbb{R}); AJA^t = J\} = f^{-1}(J)$$

for the map $f : M_{2n}(\mathbb{R}) \rightarrow M_{2n}(\mathbb{R})$ defined by $f(A) = AJA^t$.

- a. Let $T_{I_{2n}}f : M_{2n}(\mathbb{R}) \rightarrow M_{2n}(\mathbb{R})$ denote the tangent map (= differential map) of f at the identity matrix $I_{2n} \in M_{2n}(\mathbb{R})$ where, since $M_{2n}(\mathbb{R})$ is a vector space, we use the canonical identification between the tangent space $T_{I_{2n}}M_{2n}(\mathbb{R})$ and $M_{2n}(\mathbb{R})$. Determine the dimension of the image of $T_{I_{2n}}f$.
- b. Show that there is a neighborhood U of I_{2n} in $M_{2n}(\mathbb{R})$ such that $\Sigma_n \cap U$ is a submanifold of $M_{2n}(\mathbb{R}) \cong \mathbb{R}^{4n^2}$. What is its dimension?

Problem 4. Let M be an m -dimensional submanifold of \mathbb{R}^n . Show that, if $n > 2m + 1$, there exists a hyperplane $H \subset \mathbb{R}^n$ such that the restriction of the orthogonal projection $\pi_H : \mathbb{R}^n \rightarrow H$ to M is injective. Possible hint: consider the map which associates the vector $\overrightarrow{PQ} / \|\overrightarrow{PQ}\|$ to each pair $(P, Q) \in M \times M$ with $P \neq Q$.

Problem 5. Let M be a compact, oriented, smooth manifold with boundary ∂M . Prove that there does not exist a map $F : M \rightarrow \partial M$ such that $F|_{\partial M} : \partial M \rightarrow \partial M$ is the identity map.

Possible hint: consider the inclusion map $i : \partial M \rightarrow M$, and use Stokes's theorem.

Problem 6. Let $S^n \subset \mathbb{R}^{n+1}$ denote the unit sphere, and let $\omega \in \Omega^n(S^n)$ be the differential n -form given by the restriction of $x_{n+1}dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(\mathbb{R}^{n+1})$ to S^n . Show that the class of ω in the de Rham cohomology $H_{\text{dR}}^n(S^n)$ is nontrivial, namely that ω is closed but not exact.

Geometry and Topology Graduate Exam
Spring 2019

Solve all 6 problems. Partial credit will be given to partial solutions.

Problem 1. Let $X = S^2 / \sim$ be the quotient of the sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$$

by the equivalence relation \sim that glues together the three points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$; namely, one equivalence class of \sim is equal to $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, and all other equivalence classes consist of single points. Compute the fundamental group $\pi_1(X; x_0)$ for your preferred choice of base point $x_0 \in X$.

Problem 2.

Recall that the wedge sum $Y \vee Z$ of two spaces Y and Z , each equipped with a base point y_0 and z_0 , is obtained from the disjoint union $Y \amalg Z$ by gluing $x_0 \in X$ to $y_0 \in Y$. Let $X = S^1 \vee S^2$ be the wedge sum of the circle S^1 and the sphere S^2 (for arbitrary choices of base points).

- a. Draw a picture of the universal cover \tilde{X} of X .
- b. Compute the homology group $H_2(\tilde{X}; \mathbb{Z})$, with integer coefficients.

Problem 3. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be defined by $f(z) = 2z^3 + 3z^2$. Note that $f^{-1}(\{0, 1\}) = \{-\frac{3}{2}, -1, 0, \frac{1}{2}\}$ (no need to check this).

- a. Show that the restriction $g: \mathbb{C} - \{-\frac{3}{2}, -1, 0, \frac{1}{2}\} \rightarrow \mathbb{C} - \{0, 1\}$ of f is a covering map. Hint: first show that g is a local diffeomorphism.
- b. What is the index of the subgroup $g_*(\pi_1(\mathbb{C} - \{-\frac{3}{2}, -1, 0, \frac{1}{2}\}; 1))$ in the fundamental group $\pi_1(\mathbb{C} - \{0, 1\}; 5)$?

Problem 4.

Let M be a smooth m -dimensional submanifold of \mathbb{R}^n , and let

$$S_r^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n; \sum_{i=1}^n x_i^2 = r^2\}$$

denote the sphere of radius r centered at the origin in \mathbb{R}^n . Show that, for every $\varepsilon > 0$, there exists an r in the interval $[1 - \varepsilon, 1 + \varepsilon]$ such that the intersection $M \cap S_r$ is a submanifold of M of dimension $m - 1$. Possible hint: consider the map $f: M \rightarrow \mathbb{R}$ defined by $f(x_1, \dots, x_n) = \sum_{i=1}^n x_i^2$.

Problem 5.

Let $M = \{(x, y, z, w) \in \mathbb{R}^4; x^2 + y^2 + z^2 - w^4 = -1\}$.

- a. Prove that M is a differentiable submanifold of \mathbb{R}^4 .
- b. Let f be the map $\mathbb{R}^4 \rightarrow \mathbb{R}$ sending $(x, y, z, w) \mapsto w$. Compute the critical values of the restriction $f|_M: M \rightarrow \mathbb{R}$. Possible hint: the tangent map $T_p f|_M$ of the restriction $f|_M$ at $p \in M$ is the restriction of $T_p f$ to $T_p M = \ker T_p g$ where $g: \mathbb{R}^4 \rightarrow \mathbb{R}$ is defined by $g(x, y, z, w) = x^2 + y^2 + z^2 - w^4$.

Problem 6. Let Z be the vector field on \mathbb{R}^2 defined by $Z(x, y) = -y \frac{\partial}{\partial y} + 2x \frac{\partial}{\partial x}$. Compute the Lie derivative $\mathcal{L}_X(dx \wedge dy)$.

Geometry and Topology Graduate Exam
Fall 2019

Solve all seven problems. Every problem is weighted equally. Partial credit will be given to partial solutions.

Problem 1. Let X be the union of the twelve edges of a regular cube in \mathbb{R}^3 . Compute the fundamental group of the complement $\mathbb{R}^3 - X$.

Problem 2. Let $\mathbb{C}\mathbb{P}^1$ denote the complex projective space of dimension 1, and let $f: \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ be the map induced by the polynomial

$$P(X) = X^7 + 5X^3 - 6X^2 + 1.$$

(So, f sends the line passing through $(x, 1) \in \mathbb{C}^2$ to the line passing through $(P(x), 1)$.) If $\alpha \in \Omega^2(\mathbb{C}\mathbb{P}^1)$ is a differential form of degree 2 and we write

$$K := \int_{\mathbb{C}\mathbb{P}^1} \alpha,$$

compute the integral

$$\int_{\mathbb{C}\mathbb{P}^1} \Omega^2(f)(\alpha)$$

of the pullback $\Omega^2(f)(\alpha) \in \Omega^2(\mathbb{C}\mathbb{P}^1)$ of α under f in terms of K . (Possible hint: degree.)

Problem 3. Let X and Y be two topological spaces and let $f, g: X \rightarrow Y$ be two continuous maps. Consider the topological space

$$Z = \left(Y \sqcup (X \times [0, 1]) \right) / \begin{array}{l} (x, 0) \sim f(x) \\ (x, 1) \sim g(x) \end{array}$$

obtained from the disjoint union $Y \sqcup (X \times [0, 1])$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form

$$\cdots \longrightarrow H_{n+1}(Z) \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(Z) \longrightarrow H_{n-1}(X) \longrightarrow \cdots,$$

and identify the homomorphisms involved.

Problem 4. Let $f: S^n \rightarrow S^n$ be a continuous map that has no fixed points. Find the degree of f . (Hint: $a(x) = -x$.)

Problem 5. Let $p \in \mathbb{R}[x_1, \dots, x_n]$ be a nonzero polynomial over \mathbb{R} in n variables that is homogenous of degree d (i.e. $p(\lambda \cdot \vec{x}) = \lambda^d \cdot p(\vec{x})$ for all $\lambda \in \mathbb{R}$). Show that $p^{-1}(c)$ is a submanifold of \mathbb{R}^n for all $c \neq 0$.

Problem 6. Define a differential 1-form on the cylinder

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \subset \mathbb{R}^3$$

which is closed but not exact (and show that it has these properties).

Problem 7. Prove that the subset

$$M = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1^2 + x_2^2 + x_3^2 = x_4^3 + x_5^3\}$$

is not a submanifold of \mathbb{R}^5 . (Hint: $(0, 0, 0, 0, 0)$.)

Geometry and Topology Graduate Exam
Spring 2020

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Show that the subset

$$X = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5; x_1^4 + x_2^4 = 1 + x_3^2 + x_4^2 + x_5^2\} \subset \mathbb{R}^5$$

is an orientable manifold of dimension 4. Make sure that you justify the word “orientable”.

Problem 2. Let M be a compact orientable manifold with boundary. Show that there is no differentiable map $f: M \rightarrow \partial M$ such that $f(x) = x$ for every $x \in \partial M$.

Possible hint: Consider a volume form on ∂M .

Problem 3. Let M be an m -dimensional submanifold of \mathbb{R}^{2m+2} . Show that there is a hyperplane $H \subset \mathbb{R}^{2m+2}$ such that the restriction to M of the orthogonal projection $\pi_H: \mathbb{R}^n \rightarrow H$ is injective.

Possible hint: Use a suitable map $\{(x, y) \in M \times M; x \neq y\} \rightarrow S^{2m+1}$.

Problem 4. Consider the space X obtained from the cylinder $S^1 \times [0, 1]$ by identifying the antipodal points of the circle $S^1 \times \{0\}$ and identifying the antipodal points of the circle $S^1 \times \{1\}$. (Recall that the antipodal point of $(x, y) \in S^1 \subset \mathbb{R}^2$ is the point $(-x, -y)$.) Compute the fundamental group of X .

Problem 5. Use the Mayer–Vietoris exact sequence to compute, for any space X , the homology groups $H_p(X \times S^n)$ in terms of the homology groups $H_q(X)$ (for any coefficients).

Problem 6. Let $X = S^1 \vee S^1$ be the wedge sum of two circles, namely the union of two circles meeting in exactly one point x_0 . Let a_1, a_2 be the generators of $\pi_1(X; x_0)$ represented by loops going around the first and second circles, respectively. By covering space theory, for every subgroup H of $\pi_1(X; x_0)$ there is a connected covering space $p: \tilde{X} \rightarrow X$ such that the image of $p_*: \pi_1(\tilde{X}; \tilde{x}_0) \rightarrow \pi_1(X; x_0)$ is equal to H . Draw a picture of \tilde{X} when H is the subgroup generated by the element $a_1 a_2 a_1 a_2 \in \pi_1(X; x_0)$.

Problem 7. For the n -dimensional sphere S^n , consider the inclusion map $i: S^n \rightarrow \mathbb{R}^{n+1} - \{0\}$ and a closed differential form $\omega \in \Omega^n(\mathbb{R}^{n+1} - \{0\})$. Let M be a compact oriented n -dimensional manifold without boundary and, for a differentiable map $f: M \rightarrow \mathbb{R}^{n+1} - \{0\}$, let $f^*(\omega) = \Omega^n(f)(\omega) \in \Omega^n(M)$ denote the pullback of ω under f . Show that the integral $\int_M f^*(\omega)$ is an integer multiple of $\int_{S^n} i^*(\omega)$.

Possible hint: Consider the radial projection $p: \mathbb{R}^{n+1} - \{0\} \rightarrow S^n$.

Geometry and Topology Graduate Exam
Fall 2020

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Suppose that the space X admits a universal covering space \tilde{X} , namely a covering space that is path connected and simply connected. Show that, if \tilde{X} is compact, the fundamental group of X is finite.

Problem 2. Let X be the topological space obtained from a regular $2n$ -gon by identifying opposite edges with parallel orientations. Write a presentation for its fundamental group $\pi_1(X; x_0)$, for a base point $x_0 \in X$ of your choice. (The answer, as a function of n , may depend on the parity of n .)

Problem 3. In $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, consider the subset

$$X = (S^1 \times S^1) \cup (\{x_0\} \times B^2) \cup (B^2 \times \{x_0\})$$

where the disk B^2 is bounded by the unit circle S^1 in \mathbb{R}^2 , and where $x_0 \in S^1$. Compute the homology group $H_p(X; \mathbb{Z})$ for all p .

Problem 4. In the vector space $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ of all n -by- n matrices, let $SL_n(\mathbb{R})$ be the special linear group consisting of all $A \in M_n(\mathbb{R})$ with $\det A = 1$. For $A \in SL_n(\mathbb{R})$, describe the tangent space $T_A SL_n(\mathbb{R}) \subset M_n(\mathbb{R})$ by an explicit equation. Possible hint: begin with the case where A is the identity matrix I_n .

Problem 5. Consider the 1-form $\lambda \in \Omega^1(M)$ and the 2-form $\omega = d\lambda \in \Omega^2(M)$ on the manifold M . Suppose that L is a submanifold of M and that, for the inclusion map $i: L \rightarrow M$, the pull-back $i^*(\lambda) \in \Omega^1(L)$ is exact, in the sense that there exists a function $\varphi: L \rightarrow \mathbb{R}$ such that $i^*(\lambda) = d\varphi$. Show that, for the unit disk $D^2 \subset \mathbb{R}^2$ and for any smooth map $f: D^2 \rightarrow M$ which sends the boundary of the disc to L ,

$$\int_{D^2} f^*(\omega) = 0.$$

Problem 6. Let $f: S^1 \times S^1 \rightarrow S^1 \times S^1$ be the map that, identifying S^1 with the unit circle in the complex plane \mathbb{C} , is defined by

$$f(z_1, z_2) = (z_1^2 z_2, z_1^{-1} z_2)$$

for every $z_1, z_2 \in S^1 \subset \mathbb{C}$. Compute the homomorphism $H^2(f): H_{\text{dR}}^2(S^1 \times S^1) \rightarrow H_{\text{dR}}^2(S^1 \times S^1)$ induced by f on the de Rham cohomology space $H_{\text{dR}}^2(S^1 \times S^1)$. Hint: what is the degree of f ?

Problem 7. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ be a 2-form on \mathbb{R}^4 with standard coordinates x_1, x_2, x_3, x_4 . Consider the vector field $Z = 3x_1 \partial_{x_1} + 3x_2 \partial_{x_2} + 3x_3 \partial_{x_3} + 3x_4 \partial_{x_4}$, and let $(\varphi_t)_{t \in \mathbb{R}}$ be the flow that it defines; you may take for granted that this flow exists for all time t . Calculate the pull back $(\varphi_t)^* \omega$. Hint: look at the differential equation that $(\varphi_t)^* \omega$ satisfies.

Geometry and Topology Graduate Exam
Spring 2021

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Let $X = S^1 \times S^1 - \{p, q\}$, with $p \neq q$, be the twice punctured 2-dimensional torus.

- (1) Compute the homology groups $H_n(X, \mathbb{Z})$.
- (2) Compute the fundamental group of X .

Problem 2. Let X be the figure eight, union of two circles meeting in exactly one point x_0 . Recall that the fundamental group $\pi_1(X; x_0)$ is the free group on two generators a and b , respectively going once around the first and the second circle. Draw a covering $p: \tilde{X} \rightarrow X$ such that \tilde{X} is connected and $p_*(\pi_1(\tilde{X}; \tilde{x}_0))$ is the subgroup $G \subset \pi_1(X; x_0)$ generated by the subset $\{a^2, b^2, aba, bab\}$.

Use this construction to decide whether this subgroup G is normal or not.

Problem 3. Let M be a differentiable (not necessarily orientable) manifold. Show that its cotangent bundle

$$T^*M = \{(x, u); x \in M \text{ and } u: T_x M \rightarrow \mathbb{R} \text{ linear}\}$$

is a manifold, and is orientable.

Problem 4.

Show that, if a map $f: S^n \rightarrow S^n$ has no fixed points, then its degree is equal to $(-1)^{n+1}$. Possible hint: Show that f is homotopic to a simple map.

Problem 5. Let T be the torus in \mathbb{R}^3 obtained by revolving the circle

$$\{(x, y, z) \in \mathbb{R}^3; (x-2)^2 + y^2 = 1 \text{ and } z = 0\}$$

around the y -axis. Compute the integral

$$\int_T xdy \wedge dz - ydx \wedge dz + zdx \wedge dy.$$

Problem 6. Let $f: M \rightarrow N$ be a differentiable map between two connected compact orientable manifolds of the same dimension n . Suppose that there exists a nonempty open subset U such that $f^{-1}(U)$ can be written as a disjoint union $U_1 \amalg U_2 \amalg U_3$ for which each restriction $f|_{U_i}: U_i \rightarrow U$ is a diffeomorphism. Show that f is necessarily surjective.

Problem 7. Consider the differential 2-form $\omega = \frac{dx \wedge dy}{x^2 + y^2}$ on $X = \mathbb{R}^2 - \{0\}$, and denote by $Y = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$ the unit circle inside X . Prove that, for the unit disk D^2 and for any smooth map $f: D^2 \rightarrow X$ which sends the boundary of the disc to Y ,

$$\int_{D^2} f^*(\omega) = 0,$$

where $f^*(\omega) \in \Omega^2(D^2)$ (also denoted as $\Omega^2(f)(\omega)$) is the pull back of ω under f .

Geometry and Topology Graduate Exam
Fall 2021

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Find all of the 2-sheeted covering spaces (connected or disconnected) of $S^1 \times S^1$, up to isomorphism of covering spaces without basepoints.

Problem 2. Let $f : \mathbb{R}^4 \rightarrow \mathbb{R}$ be the function defined by

$$f(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

- (a) Find a real number r such that $f^{-1}(r)$ is a smooth manifold and prove it.
- (b) Find a real number r such that $f^{-1}(r)$ is not a smooth manifold and prove it.

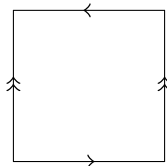
Problem 3. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, and $i : S^2 \rightarrow \mathbb{R}^3$ be the inclusion. Compute the integral over S^2 of the restriction

$$\int_{S^2} \omega = \int_{S^2} i^* \omega$$

of the 2-form on \mathbb{R}^3 given by $\omega = 2x^2 dx \wedge dz - x dy \wedge dz + 3y dx \wedge dz$.

Problem 4. Let \mathcal{D} be the distribution on $\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R} = \{(x, y, z) \in \mathbb{R}^3 | y > 0\}$ given by the kernel of the 1-form $\alpha = dz - \log(y)dx$. Is \mathcal{D} integrable? Provide justification.

Problem 5. Let K be the Klein bottle (the closed square with boundary identifications as pictured below).



- (a) Let $p \in K$ be the image of some point in the interior of the closed square (under the identifications above). Say whether the following assertion is true or false (and give justification): $K \setminus \{p\}$ is homotopy equivalent to $S^1 \vee S^1$.
- (b) Show that K is homeomorphic to the disjoint union of two Möbius bands with the boundary circles identified.
- (c) Use part (b) (whether or not you solved it) to compute $\pi_1(K)$ via van Kampen's theorem and the integral singular homology $H_*(K; \mathbb{Z})$ via the Mayer-Vietoris long-exact sequence.

Problem 6. A *space-filling curve* is a continuous surjective map $f : \mathbb{R} \rightarrow \mathbb{R}^2$ (it is a classical fact that such curves exist).

- (a) Prove that if f is any such space-filling curve, then f cannot be smooth. Equivalently, prove that if $f : \mathbb{R} \rightarrow \mathbb{R}^2$ is any smooth map, then f cannot be surjective.
- (b) Prove that if f is any space-filling curve, then f cannot be a homeomorphism.

Problem 7. Let X be the space given by taking the circle S^1 and attaching two 2-cells to S^1 along degree 9 and 12 attaching maps respectively, and then identifying a point in the interior of the first 2-cell with a point in the interior of the second 2-cell. Compute the integral homology $H_*(X; \mathbb{Z})$ in every degree.

Geometry and Topology Graduate Exam
Spring 2022

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Let B^3 be the solid ball of radius 2 centered at the origin in \mathbb{R}^3 . Let S^1 be the unit circle in the xy -plane. Compute the homology groups of $B^3 - S^1$.

Problem 2. Prove that the fundamental group and the homology groups (in every degree) of $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ are isomorphic, but that the homology groups of their universal covering spaces are not.

Problem 3. Consider the vector fields $v_k = x^k \frac{\partial}{\partial x}$ on \mathbb{R} , where $k \geq 0$.

- (1) Find the Lie bracket $[v_i, v_j]$.
- (2) Do the flows of v_i and v_j commute?
- (3) Find the flow of v_2 .
- (4) Is v_2 complete on \mathbb{R} , i.e. does every one of its flow curves exist for all time?

Problem 4. (*intersecting with a plane and a cylinder*) Let M^m be any submanifold of \mathbb{R}^N with $N \geq 3$. Show that there exist real numbers $a, b > 0$ so that the intersection $M \cap \{x_N = a\} \cap \{x_1^2 + x_2^2 = b^2\}$ is a submanifold (of M and hence \mathbb{R}^N) of dimension $m - 2$. **Note:** The condition $N \geq 3$ is not strictly necessary, it is imposed simply to ensure that $x_N, x_1,$ and x_2 are all different coordinates to avoid the case $\{x_N = a\} \cap \{x_1^2 + x_2^2 = b^2\}$ is empty (in which case the result would be vacuously still true).

Problem 5.

Say whether each assertion is true or false, with complete justification.

- (1) $S^3 \setminus \{3 \text{ points}\}$ is homotopy equivalent to $S^2 \setminus \{2 \text{ points}\}$.
- (2) Any continuous map $S^2 \rightarrow S^1 \times S^1$ is null homotopic.

Problem 6. Let $f : M^n \rightarrow N^n$ be a smooth map between compact oriented manifolds of the same dimension $n > 1$, and suppose f factors as $M^n \rightarrow S^1 \times \mathbb{R}^{n-1} \rightarrow N^n$ (with $n - 1 > 0$). Show that in any neighborhood U of any point $x \in N^n$, there exists a point $y \in U$ with $f^{-1}(y)$ having an even number of points.

Problem 7. Let M^m be a manifold and $\lambda \in \Omega^1(M)$ a closed 1-form ($d\lambda = 0$). Suppose $f : S^2 \rightarrow M$ is a smooth map, where S^2 is the standard unit 2-sphere in \mathbb{R}^3 , and let $f|_{S^1}$ denote the restriction of f to the equator circle $S^1 = \{z = 0\} \cap S^2 \subset S^2$. Show that

$$\int_{S^1} (f|_{S^1})^* \lambda = 0.$$

Geometry and Topology Graduate Exam

Fall 2022

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Consider the vector fields $X = e^x \partial_x$ and $Y = \partial_y$ on \mathbb{R}^2 . Find all vector fields Z on \mathbb{R}^2 such that $[X, Z] = [Y, Z] = 0$.

Problem 2. Let X be a path-connected topological space, and let $x \in X$. Show that $\pi_1(X, x)$ is trivial if and only if, for any $x_1, x_2 \in X$, any two paths $\gamma, \delta: [0, 1] \rightarrow X$ from x_1 to x_2 are homotopic (through paths from x_1 to x_2).

Problem 3. Let $\mathbb{T}^2 = S^1 \times S^1$ be the 2-torus, and let $\alpha, \beta, \gamma \in \Omega^1(\mathbb{T}^2)$ be closed 1-forms on \mathbb{T}^2 . Show that there exist real numbers $a, b, c \in \mathbb{R}$ such that $a\alpha + b\beta + c\gamma$ is exact.

Problem 4. Let ω be a 1-form on the sphere $S^2 = \{(x, y, z) | x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Show that if ω is invariant under rotations, i.e. $\phi^* \omega = \omega$ for all $\phi \in SO(3)$, then $\omega = 0$.

Problem 5. Show that if M and N are compact, connected smooth manifolds, then every submersion $f: M \rightarrow N$ is surjective.

Problem 6. Let X be the complement of a point in $S^1 \times S^1 \times S^1$. Calculate the fundamental group and homology groups of X .

Problem 7. Show that every continuous map from $\mathbb{R}P^2 \times \mathbb{R}P^2$ to $S^1 \times S^1 \times S^1 \times S^1$ is null-homotopic.

Geometry and Topology Graduate Exam
Spring 2023

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Let X be a Hausdorff topological space, and let $\pi : \tilde{X} \rightarrow X$ be its universal cover, i.e. \tilde{X} is path connected and simply connected and π is a covering map. Prove that if \tilde{X} is compact then the fundamental group of X is finite.

Problem 2. Let A be an $n \times n$ matrix which is symmetric and nonsingular, and let c be a nonzero real number. Prove that

$$\{x \in \mathbb{R}^n \mid \langle Ax, x \rangle = c\}$$

is a smooth submanifold of \mathbb{R}^n , and state its dimension. Here $\langle -, - \rangle$ denotes the standard inner product on \mathbb{R}^n .

Problem 3. Let $\omega \in \Omega^2(M)$ be an exact 2-form on a manifold M . Prove that for any map $f: S \rightarrow M$ from a closed orientable surface (i.e., closed orientable 2-dimensional manifold) S , there must be some $p \in S$ such that $(f^*\omega)_p = 0$.

Problem 4. Let $\mathbb{T}^2 = S^1 \times S^1$ denote the standard 2-torus and S^2 the standard 2-sphere. Let X be the space obtained by identifying two distinct points a_1, a_2 from \mathbb{T}^2 to some point $p \in S^2$. Compute (1) the (integral) homology groups of X in every degree, and (2) the fundamental group of X .

Problem 5. Let $n > 1$, let $\mathbb{T}^n = (S^1)^n$ denote the n -torus, and let S^n denote the standard unit sphere in \mathbb{R}^{n+1} .

- (a) Let $f: \mathbb{T}^n \rightarrow S^n$ be a smooth map satisfying the following properties:
- there exists 5 mutually disjoint open subsets U_1, \dots, U_5 of \mathbb{T}^n such that for each i $f|_{U_i}$ is a diffeomorphism from U_i onto the open southern hemisphere $S^n \cap \{x_{n+1} < 0\}$;
 - The image of the complement of these subsets lies in the northern hemisphere; that is $f(\mathbb{T}^n - \cup_i U_i) \subset S^n \cap \{x_{n+1} \geq 0\}$.

(You may take for granted such an f exists). Show that the induced map on n th de Rham cohomology $f^*: H_{dR}^n(S^n) \rightarrow H_{dR}^n(\mathbb{T}^n)$ must be non-zero.

- (b) Show that there does *not* exist a continuous map $f: S^n \rightarrow \mathbb{T}^n$ from the n -sphere to the n -torus $\mathbb{T}^n = (S^1)^n$ inducing a non-zero map $f_*: H_n(S^n) \rightarrow H_n(\mathbb{T}^n)$ of n -th homology groups.

Problem 6. Let X be a topological space. Suppose for some k that we can cover X by k open sets U_1, \dots, U_k so that each U_i is contractible as is each higher intersection of s open sets $U_{i_1} \cap \dots \cap U_{i_s}$ for every s . Prove that the reduced homology $\tilde{H}_i(X) = 0$ for all $i \geq k - 1$.

Problem 7. Let X denote the vector field on \mathbb{R}^3 given in standard coordinates by $X = x_1 \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_2} + 3x_3 \frac{\partial}{\partial x_3}$, and let $\phi_t: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ denote the induced flow (you may take for granted that this exists for all time and is an oriented diffeomorphism).

If $R = [0, 1]^3$ denotes the unit cube, compute the rate of change of the (standard) volume of $\phi_t(R)$ at $t = 0$. That is, compute:

$$\frac{d}{dt} \left(\int_{\phi_t(R)} dx_1 dx_2 dx_3 \right)_{t=0}.$$

Hint: Re-express the above integral as an integral of a form which is varying in t , over a region that is not varying in t . You may also use the fact that $\frac{d}{dt} \int_A (\omega_t) = \int_A \frac{d}{dt} (\omega_t)$.

Geometry and Topology Graduate Exam
Fall 2023

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Show that for any $n \geq 1$, the subset $SL_n(\mathbb{R}) \subset \mathbb{R}^{n \times n}$ consisting of matrices of determinant 1 is a smooth submanifold.

Problem 2. Let $\alpha \in \Omega^2(\mathbb{R}^3)$ be the 2-form defined by

$$\alpha = (x^3 + y + z) dy \wedge dz - (x + y^3 + z) dx \wedge dz + (x + y + z^3) dx \wedge dy.$$

Compute the integral

$$\int_{S^2} \alpha$$

over the unit sphere $S^2 \subset \mathbb{R}^3$ (endowed with a fixed orientation of your choosing).

Hint: Recall that, for the spherical coordinates (r, θ, φ) with $x = r \cos \theta \cos \varphi$, $y = r \sin \theta \cos \varphi$, and $z = r \sin \varphi$, we have $dx \wedge dy \wedge dz = r^2 \cos \varphi dr \wedge d\theta \wedge d\varphi$.

Problem 3. Let X be a manifold with $\pi_2(X, x) = 0$ for all $x \in X$. Is it necessarily the case that $H_2(X; \mathbb{Z}) = 0$ as well? Explain.

Problem 4. What are the integral homology groups of $S^1 \vee S^2 \vee S^3 \vee S^4$?

Problem 5. Let V be a smooth vector-field on a closed manifold M and let $\varphi : \mathbb{R} \times M \rightarrow M$ be the flow generated by V . Consider the quotient space X by the flow, i.e.

$$X = M/\sim \quad \text{where} \quad p \sim q \text{ if and only if } q = \varphi(t, p) \text{ for some time } t.$$

Prove or give a counter-example to the following statements.

- (a) X is compact.
- (b) X is a closed manifold.

Problem 6. Give an example of a covering space $X \rightarrow Y$ which is not a regular covering space.

Problem 7. Show that the Cantor set does not admit a CW complex structure.

Geometry and Topology Screening Exam
Spring 2024

Instructions: Solve as many of the following 7 problems as you can. Solutions will be graded for correctness, completeness and clarity. Partial credit will be awarded for partial solutions indicating clear progress.

Problem 1. Let M be a path-connected smooth manifold. Prove that for any $p_1, p_2 \in M$, there is a diffeomorphism φ of M with $\varphi(p_1) = p_2$.

Problem 2. Show that for any vector bundle $p: E \rightarrow B$ over a space B , the map p is a homotopy equivalence.

Problem 3. Let X be a closed manifold, and let θ be a degree 1 de Rham class such that

$$\int_{\Gamma} \theta \in \mathbb{Z} \quad \text{for any cycle } \Gamma.$$

Show that there exists a map $\varphi: X \rightarrow S^1$ so that the induced map

$$\varphi^*: H_{\text{dR}}^1(S^1) \rightarrow H_{\text{dR}}^1(X)$$

contains θ in its image.

Problem 4. Suppose $f: M \rightarrow N$ is a smooth map between connected smooth manifolds which is both an immersion and a submersion. Is f necessarily a diffeomorphism? Prove it or provide a counter-example.

Problem 5. Let 0_n be the $n \times n$ 0-matrix and let I_n be the $n \times n$ identity matrix. Let J be the $2n \times 2n$ matrix given by

$$J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$$

Finally, let $G \subset \text{GL}_{2n}(\mathbb{R})$ be the group of invertible $2n \times 2n$ matrices A that commute with J . Show that G is a sub-manifold of $\text{GL}_{2n}(\mathbb{R})$ and compute its dimension.

Problem 6. Let X be a closed smooth manifold and let V be a vector-field. Suppose that α and β are closed 1-forms on X such that $\alpha(V)$ and $\beta(V)$ are constant functions. Show that

$$\alpha \wedge \beta$$

is invariant under the flow generated by V .

Problem 7. Let $K \subset S^3$ be a smoothly embedded S^1 . Compute the first homology group of $S^3 \setminus K$ over \mathbb{Z} .

Topology qualifying exam Fall 2024

1. Let $\mathbb{R}P^2 \vee \mathbb{R}P^2$ denote the wedge sum of two copies of $\mathbb{R}P^2$. Explicitly, picking a basepoint $x_0 \in \mathbb{R}P^2$, we put

$$\mathbb{R}P^2 \vee \mathbb{R}P^2 := (\mathbb{R}P^2 \times \{x_0\}) \cup (\{x_0\} \times \mathbb{R}P^2) \subset \mathbb{R}P^2 \times \mathbb{R}P^2.$$

- (a) Compute the fundamental groups of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ and $\mathbb{R}P^2 \times \mathbb{R}P^2$.
- (b) Prove that $\mathbb{R}P^2 \vee \mathbb{R}P^2$ is not a retract of $\mathbb{R}P^2 \times \mathbb{R}P^2$.
- (c) Prove that any map $\mathbb{R}P^2 \vee \mathbb{R}P^2 \rightarrow S^1$ is nullhomotopic.
- (d) Give an example of a map $\mathbb{R}P^2 \times \mathbb{R}P^2 \rightarrow \mathbb{R}P^2 \vee \mathbb{R}P^2$ which is not nullhomotopic.
2. Let $f : S^{2n-1} \rightarrow S^n$ be a smooth map, and let $\lambda \in \Omega^n(S^n)$ be an n -form with $\int_{S^n} \lambda = 1$.
- (a) Show that there exists an $\omega \in \Omega^{n-1}(S^{2n-1})$ such that $d\omega = f^*(\lambda)$.
- (b) Show that if $\omega' \in \Omega^{n-1}(S^{2n-1})$ is another form with $d\omega' = f^*(\lambda)$, then $\int_{S^{2n-1}} \omega \wedge f^*(\lambda) = \int_{S^{2n-1}} \omega' \wedge f^*(\lambda)$.
3. (a) Give an example of a path-connected topological space X whose fundamental group is nonzero and isomorphic to $H_1(X)$.
- (b) Let X be a connected CW complex and $A \subset X$ a subcomplex such that A is homotopy equivalent to S^3 and X/A is homotopy equivalent to S^5 . Compute $H_n(X)$ for all n .
- (c) Give an example of a topological space X whose reduced homology groups are $\tilde{H}_5(X) \cong \mathbb{Z}$, $\tilde{H}_2(X) \cong \mathbb{Z}/2\mathbb{Z}$, and $\tilde{H}_n(X) \cong 0$ for $n \neq 2, 5$.
4. Let G be a topological group. This means G is a group which is also equipped with a topology such that the multiplication map $G \times G \rightarrow G$, $(g, h) \mapsto g \cdot h$ and inversion map $G \rightarrow G$, $g \mapsto g^{-1}$ are both continuous. Assuming that G is connected, prove that the fundamental group of G is abelian.

Topology qualifying exam **Spring 2025**

1. Let X be the quotient S^2/\sim by the relation

$$(x, y, z) \sim (x', y', z') \quad \text{if and only if} \quad z = z' = 0 \text{ and } (x, y) = \pm(x', y')$$

In other words, X is acquired by S^2 by quotienting the equator $S^1 \subset S^2$ by the map $v \rightarrow -v$.

- (a) Describe a cell decomposition of X . That is, describe the cells and the attaching maps.
 - (b) Give the corresponding cellular homology complex (with \mathbb{Z} -coefficients) for the cell decomposition in (a).
 - (c) Compute the homology of the cellular complex from (b).
2. Consider the space $Y = \mathbb{R}\mathbb{P}^3 \vee (S^2 \times S^1)$.
- (a) Compute the homology groups (with \mathbb{Z} coefficients) of Y .
 - (b) Is Y homotopy equivalent to a compact orientable manifold? Prove or disprove.
 - (c) Compute the fundamental group $\pi_1(Y)$
 - (d) Compute the Euler characteristic $\chi(Y)$.
3. (a) Let ΣX denote the suspension of a connected topological space X .¹ Using the Mayer–Vietoris sequence, compute the integral homology groups of ΣX in terms of those of X .
- (b) Prove that there is an isomorphism $\pi_7(S^3) \cong \pi_7(S^2)$.
4. (a) Let X_n be the complement of n distinct lines through the origin in \mathbb{R}^3 . Compute $\pi_1(X_n)$.

¹Recall that the suspension of a topological space X is given by $\Sigma X := (X \times I)/\sim$, where we put $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all $x, y \in X$.

- (b) Give a pair of path-connected topological spaces X and Y such that $H_1(X)$ and $H_1(Y)$ are isomorphic, but $\pi_1(X)$ and $\pi_1(Y)$ are not.
- (c) Compute the fundamental group of a genus g surfaces with n points removed for $g, n > 0$.
- (d) Let X be a path-connected space such that $\pi_1(X) \cong \mathbb{Z}/7\mathbb{Z}$. Determine the set of integers n such that there exists a 9-sheeted covering space $p : \tilde{X} \rightarrow X$ with n connected components.