Spring 2012 Real Analysis Exam

Answer all four questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let f and g be real integrable functions on a σ-finite measure space $(X, \mathcal{M}, \mu)$, and for $t \in \mathbb{R}$ let

$$F_t = \{x \in E : f(x) > t\} \quad \text{and} \quad G_t = \{x \in E : g(x) > t\}.$$

Show that

$$\int_X |f-g| d\mu = \int_{-\infty}^{\infty} \mu\left((F_t \backslash G_t) \cup (G_t \backslash F_t)\right) dt.$$

2. Show that

$$\int_{\pi}^{\infty} \frac{dx}{x^2 (\sin^2 x)^{1/3}}$$

is finite.

3. A collection of functions $\{f_\alpha\}_{\alpha \in \mathcal{A}} \subset L^1(\mu)$ on the measure space $(X, \mathcal{M}, \mu)$ is said to be *uniformly integrable* if

$$\lim_{M \to \infty} \sup_{\alpha \in \mathcal{A}} \int_{\{x : |f_\alpha(x)| > M\}} |f_\alpha| = 0.$$

a. Prove that if $f \in L^1$ then $\{f\}$ is uniformly integrable.

b. Prove that if $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{f_\beta\}_{\beta \in \mathcal{B}}$ are two collections of uniformly integrable functions then $\{f_\gamma\}_{\gamma \in A \cup B}$ is uniformly integrable.

c. Show that if $\mu(X) < \infty$ and $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ is uniformly integrable then

$$\sup_{\alpha\in\mathcal{A}} \int |f| d\mu < \infty.$$

Give an example to show that the conclusion fails without the condition $\mu(X) < \infty$.

d. Again let $\mu(X) < \infty$ and suppose $\{f_n\}_{n=0}^\infty \subset L^1(\mu)$ such that $f_n \to f_0$ a.e. and $\int |f_n| d\mu \to \int |f_0| d\mu$. Prove that $\{f_n\}_{n=0}^\infty$ is uniformly integrable. Hint: Consider some ϕ_M, a continuous, bounded function on $[0,\infty)$, equal to 0 on $[M,\infty)$, for which $|t|\mathbf{1}\{|t| > M\} \le |t| - \phi_M(|t|)$.

4. Let $\mathbb{M}$ be the collection of all finite measures on the measure space $(X, \mathcal{M})$.

a. Show that

$$d(\nu,\lambda) = 2\sup_{E\in\mathcal{M}} |\nu(E) - \lambda(E)|$$

defines a metric on $\mathbb{M}$.

b. For any $\mu \in \mathbb{M}$ that dominates measures ν and λ in $\mathbb{M}$ with $\nu(X) = \lambda(X) = 1$, let

$$p = \frac{d\nu}{d\mu} \quad \text{and} \quad q = \frac{d\lambda}{d\mu}.$$

Prove

$$d(\nu,\lambda) = \int |p(x) - q(x)|\, d\mu = 2\left(1 - \int (\min\{p(x), q(x)\})\, d\mu\right).$$

Hint: notice that $\mu(E) - \lambda(E) = \lambda(E^c) - \nu(E^c)$.

REAL ANALYSIS GRADUATE EXAM

Fall 2012

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1. Let m be the Lebesgue measure on $X = [0, 1]$. If

$$m(\limsup_{n\to\infty} A_n) = 1, m(\liminf_{n\to\infty} B_n) = 1,$$

prove that $m\left(\limsup_{n\to\infty}(A_n \cap B_n)\right) = 1$, where

$$\limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k, \liminf_{n\to\infty} B_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} B_k.$$

2. Assume $f : X \to [0, \infty)$ is measurable. Find

$$\lim_n \int_X n \log\left[1 + \frac{f(x)}{n}\right] d\mu.$$

3. Let $f \in L^1(m)$. For $k = 1, 2, \ldots$ let f_k be the step function defined by

$$\begin{aligned} f_k(x) &= k\int_{j/k}^{(j+1)/k} f(t)dt \\ \text{for } \frac{j}{k} &< x \leq \frac{j+1}{k}, j = 0, \pm 1, \ldots. \end{aligned}$$

Show that f_k converges to f in L^1 as $k \to \infty$.

4.If E is Borel set in $\mathbb{R}^n$ the density $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r\to 0} \frac{m(E \cap B(x, r))}{m(B(x, r))},$$

whenever the limit exists [Here m denotes the Lebesgue measure and $B(x, r)$ is the open ball with center at x and radius r.]

(a) Show that $D_E(x) = 0$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \notin E$.
(b) For $\alpha \in (0, 1)$ find an example of E and x such that $D_E(x) = \alpha$.
(c) Find an example of E and x such that $D_E(x)$ does not exist.

REAL ANALYSIS GRADUATE EXAM

Spring 2013

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1. Suppose that $\{f_n\}$ is a sequence of of real valued continuously differentiable functions on $[0, 1]$ such that

$$\lim_{n\to\infty} \int_0^1 |f_n(x)|\, dx = 0 \text{ and } \lim_{n\to\infty} \int_0^1 |f_n'(x)|\, dx = 0.$$

Show that $\{f_n\}$ converges to 0 uniformly on $[0, 1]$.

2. Investigate the convergence of $\sum_{n=0}^{\infty} a_n$, where

$$a_n = \int_0^1 \frac{x^n}{1-x} \sin(\pi x)\, dx.$$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $f_n, f \in L^1(\mu)$. Show that $\int_X |f_n - f|\, d\mu \to 0$ as $n \to \infty$ if and only if

$$\sup_{A\in\mathcal{M}} \left| \int_A f_n d\mu - \int_A f d\mu \right| \to 0$$

as $n \to \infty$.

4. Let μ and ν be σ-finite positive measures, $\mu \geq \nu$ and assume that $\nu << \mu - \nu$ (ν is absolutely continuous with respect to $\mu - \nu$).

Prove that

$$\mu\left(\{\frac{d\nu}{d\mu} = 1\}\right) = 0.$$

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1. Let μ be a finite Borel measure on $\mathbb{R}$, which is absolutely continuous with respect to the Lebesgue measure m. Prove that $x \mapsto \mu(A + x)$ is continuous for every Borel set $A \subseteq \mathbb{R}$.

2. Let f be a Lebesgue integrable function on $\mathbb{R}$, and assume that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty.$$

Prove that $g(x) = \sum_{n=1}^{\infty} f(a_n x)$ converges almost everywhere and is integrable on $\mathbb{R}$. Also, find an example of a Lebesgue integrable function f on $\mathbb{R}$ such that $g(x) = \sum_{n=1}^{\infty} f(nx)$ converges almost everywhere but is not integrable.

3. Assume $b > 0$. Show that the Lebesgue integral

$$\int_1^{\infty} x^{-b} e^{\sin x} \sin(2x)\, dx$$

exists if and only if $b > 1$.

4. Suppose that F is the distribution function of a Borel measure μ on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$. Prove that

$$\int_{-\infty}^{\infty} \big(F(x + a) - F(x)\big)\, dx = a$$

for all $a > 0$.

REAL ANALYSIS GRADUATE EXAM

Spring 2014

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1. Suppose that $(X, \mathcal{B}, \mu)$ is a measure space with $\mu(X) < \infty$, and that $\{f_n\}_{n\geq 1}$ and f are measurable functions on X such that $f_n \to f$ almost everywhere.

(i) Suppose that $\int f^2 \, d\mu < \infty$. Show that f is integrable.

(ii) Suppose that there exists $C < \infty$ such that $\int f_n^2 \, d\mu \leq C$ for all $n \geq 1$. Show that $f_n \to f$ in L^1.

(iii) Give an example where $\int |f_n| \, d\mu \leq 1$ for all $n \geq 1$ but $f_n \not\to f$ in L^1.

2. For what non-negative integer n and positive real c does the integral

$$\int_1^\infty \ln\left(1 + \frac{(\sin x)^n}{x^c}\right) dx$$

(a) exist as a (finite) Lebesgue integral?

(b) converge as an improper Riemann integral?

3. Suppose f is Lebesgue integrable on $\mathbb{R}$. Show that

$$\lim_{t\to 0} \int_{-\infty}^{\infty} |f(x+t) - f(x)| dx = 0.$$

4. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces such that $\mu(X) > 0$ and $\nu(Y) > 0$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be measurable functions (with respect to $\mathcal{A}$ and $\mathcal{B}$ respectively) such that

$$f(x) = g(y) \quad \mu \times \nu \text{ -almost everywhere on } X \times Y$$

Show that there exists a constant λ such that $f(x) = \lambda$ for μ-a.e. x and $g(y) = \lambda$ for ν-a.e. y.

REAL ANALYSIS GRADUATE EXAM

Fall 2014

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1. Assume that f is integrable on $(0,1)$. Prove that

$$\lim_{a\to\infty} \int_0^1 f(x)\, x\, \sin(ax^2)\, dx = 0.$$

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f and $f_1, f_2, f_3 \dots$ be real valued measurable functions on X. If $f_n \to f$ in measure and if $F\colon \mathbb{R} \to \mathbb{R}$ is uniformly continuous, prove that $F \circ f_n \to F \circ f$ in measure.

3. Let f_n be **nonnegative** measurable functions on a measure space $(X, \mathcal{M}, \mu)$ which satisfy $\int f_n \, d\mu = 1$ for all $n = 1, 2, \dots$. Prove that

$$\limsup_n \, (f_n(x))^{1/n} \leq 1$$

for μ-a.e. x.

4. Let $-\infty < a < b < \infty$. Suppose $F\colon [a,b] \to \mathbb{C}$.

(a) Define what it means for F to be absolutely continuous on $[a,b]$.

(b) Give an example of a function which is uniformly continous but not absolutely continuous. (Remember to justify your answer.)

(c) Prove that if there exists M such that $|F(x) - F(y)| \leq M|x - y|$ for all $x, y \in [a,b]$, then F is absolutely continous. Is the converse true? (Again, remember to justify your answer.)

REAL ANALYSIS GRADUATE EXAM

Fall 2015

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1. Prove that for almost all $x \in [0,1]$, there are at most finitely many rational numbers with reduced form p/q such that $q \geq 2$ and $|x - p/q| < 1/(q \log q)^2$. (Hint: Consider intervals of length $2/(q \log q)^2$ centered at rational points p/q.)

2. Suppose that the real-valued function $f(x)$ is nondecreasing on the interval $[0,1]$. Prove that there exists a sequence of continuous functions $f_n(x)$ such that $f_n \to f$ pointwise on this interval.

3. Let (X, μ) be a finite measure space. Assume that a sequence of integrable functions f_n satisfies $f_n \to f$ in measure, where f is measurable. Assume that f_n satisfies the following property: For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) \leq \delta \implies \int_E |f_n| \, d\mu \leq \epsilon.$$

Prove that f is integrable and that

$$\lim_n \int_X |f_n - f| \, d\mu = 0.$$

4. Consider the following two statements about a function $f \colon [0,1] \to \mathbb{R}$:

(i) f is continuous almost everywhere

(ii) f is equal to a continuous function g almost everywhere.

Does (i) imply (ii)? Prove or give a counterexample. Does (ii) imply (i)? Prove or give a counterexample.

REAL ANALYSIS GRADUATE EXAM

Fall 2016

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1. Let $(X, \mathcal{F}, \mu)$ be a *finite* measure space, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of *nonnegative* measurable functions. Prove that $f_n \to 0$ in measure if and only if

$$\lim_n \int \frac{f_n}{f_n + 1} d\mu = 0.$$

2. Let $(X, \mathcal{F}, \mu)$ be a *finite* measure space, and let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ be a sequence of sets. Assume that $\mu(A_n) \geq \delta$ for all $n \in \mathbb{N}$, where $\delta > 0$. Prove that there exists a set $S \in \mathcal{F}$ of positive measure such that every $x \in S$ is in A_j for infinitely many j.

3. Let $f_n \colon [0,1] \to [0,\infty)$ be Lebesgue measurable and such that $f_n(x) \to 0$ for almost every x. Assume that

$$\sup_n \int_0^1 \phi(f_n(x))\, dx \leq 1$$

for some continuous $\phi \colon [0,\infty) \to [0,\infty)$ which satisfies $\phi(t)/t \to \infty$ as $t \to \infty$. Prove that $\int_0^1 f_n(x)\, dx \to 0$ as $n \to \infty$. (Provide a detailed proof.)

4. Let $h \colon [0,\infty) \to \mathbb{R}$ be continuous with compact support. Prove that

$$\lim_{\epsilon \to 0+} \int_{\epsilon}^{\infty} \frac{h(\alpha x) - h(\beta x)}{x}\, dx = h(0) \log \frac{\alpha}{\beta}$$

for every $\alpha, \beta > 0$.

REAL ANALYSIS GRADUATE EXAM

Spring 2015

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1. Consider the sequence

$$f_n(x) = \left(1 + \frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right), \qquad n = 1, 2, \ldots .$$

Evaluate

$$\lim_n \int_0^\infty f_n(x)\, dx,$$

being careful to justify your answer.

2. Suppose that $f \colon [0, \infty) \to \mathbb{R}$ is Lebesgue integrable.
(i) Show that there exists a sequence $x_n \to \infty$ such that $f(x_n) \to 0$.
(ii) Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or a counterexample.
(iii) Suppose additionally that f is differentiable and $f'(x) \to 0$ as $x \to \infty$. Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or a counterexample.

3. Define $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$.
(i) Show that

$$\sum_{n=1}^\infty \int_0^\infty f_n(x)\, dx = 0$$

and

$$\int_0^\infty \sum_{n=1}^\infty f_n(x)\, dx = \log(b/a).$$

(ii) What can you deduce about the value of

$$\int_0^\infty \sum_{n=1}^\infty |f_n(x)|\, dx?$$

4. Assume that f is integrable on $[0, 1]$ with respect to the Lebesgue measure m, and let $F(x) = \int_0^x f(t)\, dt$. Assume that $\phi \colon \mathbb{R} \to \mathbb{R}$ is Lipschitz, i.e., there exists a constant $C \geq 0$ such that

$$|\phi(x_1) - \phi(x_2)| \leq C|x_1 - x_2|, \qquad x_1, x_2 \in \mathbb{R}.$$

Prove that there exists a function g which is integrable on $[0, 1]$ such that $\phi(F(x)) = \int_0^x g(t)\, dt$ for $x \in [0, 1]$.

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1. Let

$$f(y) = \sum_n \frac{x}{x^2 + yn^2}.$$

Show that $g(y) = \lim_{x\to\infty} f(x, y)$ exists for all $y > 0$. Find $g(y)$.

2. Let $A \subseteq \mathbb{R}$ be Lebesgue measurable. Show that $n(\chi_A * \chi_{[0,\frac{1}{n}]}) \to \chi_A$ pointwise a.e. as $n \to \infty$. (Recall that $(f * g)(x) = \int f(x - y)g(y)\, dy$ for $x \in \mathbb{R}$.)

3. a) Prove that if a sequence of integrable functions f_n on $[0, 1]$ satisfies $\int_0^1 |f_n(x)|\, dx \leq 1/n^2$ for $n \in \mathbb{N}$, then $f_n \to 0$ a.e. on $[0, 1]$ as $n \to \infty$.
b) Show that the above fact is not true if $1/n^2$ is replaced by $1/\sqrt{n}$.

4. Let

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Also, let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rationals. Define

$$g_n(x) = \frac{1}{2^n} f(x - r_n), \qquad x \in \mathbb{R}$$

and let

$$g(x) = \sum_{n=1}^{\infty} g_n(x), \qquad x \in \mathbb{R}$$

a) Prove that g is integrable on $\mathbb{R}$.
b) Prove that g is discontinuous at every point in $\mathbb{R}$.

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Spring 2017

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1. Assume that f is a positive absolutely continuous function on $[0,1]$. Prove that $1/f$ is also absolutely continuous on $[0,1]$.

2. Assume that E is Lebesgue measurable.
(i) Suppose $m(E) < \infty$, where m is the Lebesgue measure. Show that

$$f(x) = \int \chi_E(y)\chi_E(y-x)dm(y)$$

is continuous. (Here, χ_A denotes the characteristic function of a set $A \subseteq \mathbb{R}$).
(ii) Suppose $0 < m(E) \leq \infty$. Show that $S = E - E = \{x - y : x, y \in E\}$ contains an open interval $(-\epsilon, \epsilon)$ for some $\epsilon > 0$.

3. Assume that f is a continuous function on $[0,1]$. Prove that

$$\lim_{n\to\infty} \int_0^1 nx^{n-1} f(x)\,dx = f(1).$$

4. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Let f, g be measurable real valued functions. Show that

$$\int |f-g|\,d\mu = \int_{-\infty}^{\infty} \int \Big|\chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x))\Big|\,d\mu(x)dt.$$

REAL ANALYSIS GRADUATE EXAM

Fall 2017

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1. Let $(X, \mathscr{A}, \mu)$ be a measure space and f, g, f_n, g_n measurable so that $f_n \to f$ and $g_n \to g$ in measure. Is it true that $f_n^3 + g_n \to f^3 + g$ in measure if
(i) $\mu(X) = 1$
(ii) $\mu(X) = \infty$.
In both cases prove the statement or provide a counter example.

2. Let $f \in \mathscr{L}^1(\mathbb{R})$. Show that the series

$$\sum_{n=1}^{\infty} f(x+n)$$

converges absolutely for Lebesgue almost every $x \in \mathbb{R}$.

3. Assume that $E \subset \mathbb{R}$ is such that $m(E \cap (E+t)) = 0$ for all $t \neq 0$, where m is the Lebesgue measure on $\mathbb{R}$. Prove that $m(E) = 0$.

4. Let $(X, \mathscr{A}, \mu)$ be a measure space and f_n a sequence of non-negative measurable functions. Prove that if $\sup_n f_n$ is integrable then

$$\limsup_n \int_X f_n \, d\mu \leq \int_X \limsup_n f_n \, d\mu.$$

Also show that
(i) the inequality may be strict and
(ii) that the inequality may fail unless $\sup_n f_n \in \mathscr{L}^1$.

REAL ANALYSIS GRADUATE EXAM

Spring 2018

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1. Let $-\infty < a < b < \infty$ and suppose $\mathcal{B}$ is a countable collection of closed subintervals of (a, b). Give the proof that there is a countable pairwise-disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ such that

$$\bigcup_{I \in \mathcal{B}} I \subset \bigcup_{I \in \mathcal{B}'} \widetilde{I},$$

where $\widetilde{I}$ denotes the 5-times enlargment of I; thus if $I = [x - \rho, x + \rho]$ then $\widetilde{I} = [x - 5\rho, x + 5\rho]$.

2. Assume that f is absolutely continuous on $[0, 1]$, and assume that $f' = g$ a.e., where g is a continuous function. Prove that f is continuously differentiable on $[0, 1]$.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) = 1$. Let $A_1, A_2, \ldots, A_{50} \in \mathcal{M}$. Assume that almost every point in X belongs to at least 10 of these sets. Prove that at least one of the sets has measure greater than or equal to $1/5$.

4. Let $f : [0, \infty) \to \mathbb{R}$ be absolutely continuous on every closed subinterval of $[0, \infty)$ and

$$f(x) = f(0) + \int_0^x g(t)\, dt, \qquad \text{for } x \geq 0,$$

where $g \in \mathscr{L}^1([0, \infty))$. Show that

$$\int_0^\infty \frac{f(2x) - f(x)}{x}\, dx = (\log 2) \int_0^\infty g(t)\, dt.$$

REAL ANALYSIS

Fall 2018

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1. Let $f : [0, 1] \to \mathbf{R}$ be an absolutely continuous function. Let

$$g(x) = \int_0^1 f(xt)\, dt, x \in [0, 1].$$

Show that g is an absolutely continuous function.

2. Let f be Lebesgue measurable on $[0, 1]$ and assume $f(x) > 0$ for almost every x. Let E_k for $k = 1, 2, \ldots$ be a sequence of measurable sets on $[0, 1]$ so that $\int_{E_k} f(x)\, dx \to 0$ as $k \to \infty$. Show that $m(E_k) \to 0$ as $k \to \infty$.

3. Let $f \in L^1(\mathbf{R})$, and let

$$\begin{aligned} S_n(x) &= \frac{1}{n}\sum_{j=0}^{n-1} f\left(x + \frac{j}{n}\right), x \in \mathbf{R}, \\ S(x) &= \int_x^{x+1} f(y)\, dy, x \in \mathbf{R}. \end{aligned}$$

Show that $f_n \to f$ in $L^1(\mathbf{R})$.

4. Assume that f_n is a sequence of integrable functions on $\mathbf{R}$ such that

$$\lim_{n\to\infty} \int f_n(x)\, g(x)\, dx = g(0)$$

for all g continuous with compact support.

Prove that f_n is not a Cauchy sequence in $L^1(\mathbf{R})$.

REAL ANALYSIS

Spring 2019

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1. Find the limit

$$\lim_{j\to\infty}\int_{-1}^{1}\frac{1-e^{-\frac{x^2}{j}}}{x^2}dx.$$

2. Let F be absolutely continuous and $F, F' \in L^1(\mathbb{R}, m)$, where $m(dx) = dx$ is the Lebesgue measure. Prove that

$$\int_{-\infty}^{\infty} F'(x)\, dx = 0.$$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume that $h \circ f$ is integrable for every continuous function $h : \mathbb{R} \to \mathbb{R}$. Prove that there is $a > 0$ so that

$$\mu(\{x : |f(x)| > a\}) = 0.$$

4. (i) Show that for every $\varepsilon > 0$ there exists a non-negative $f \in L^1([0,1], m)$ such that $f(x) = 0$ on the set of measure $\geq 1-\varepsilon$ and

$$\int_a^b f(x)\, dx > 0$$

for all $0 < a < b < 1$.

(ii) Show that for each $\varepsilon > 0$ there exists an absolutely continuous strictly increasing h on $[0,1]$ so that $h'(x) = 0$ on a set of measure $\geq 1-\varepsilon$.

REAL ANALYSIS GRADUATE EXAM

Fall 2019

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let f be a real valued function on a set X. Let $\mathcal{M}$ be the smallest σ-algebra on X for which f is measurable. Prove that $\{x\} \in \mathcal{M}$ for all $x \in X$ if and only if f is one-to-one.

2. Let m be the Lebesgue measure on $[0, 1]$. Let $f_n \colon [0, 1] \to \mathbb{R}$ be measurable functions. Assume that $\sum_{n=1}^{\infty} \int_0^1 |f_n(x)| \, dx \leq 1$. Prove that $f_n \to 0$ as $n \to \infty$ almost everywhere.

3. Let m be the Lebesgue measure on $[0, 1]$. Prove that there does not exists a measurable set $A \subseteq [0, 1]$ such that $m(A \cap [a, b]) = (b - a)/2$ for all $0 \leq a < b \leq 1$.

4. Let $V_f(0, x)$ be the total variation of f on $[0, x]$. Prove that if $f(x)$ is absolutely continuous on $[0, 1]$, then so is $V_f(0, x)$.

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Spring 2020

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1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be strictly increasing and continuous. True or false: If $A \subseteq \mathbb{R}$ is Lebesgue measurable then $f^{-1}(A)$ is Lebesgue measurable. Provide a detailed justification of your answer.

2. Prove that the limit

$$\lim_{n\to\infty} n \int_{1/n}^{1} \frac{\cos(x + 1/n) - \cos x}{x}\, dx$$

exists. (Above, consider $n \in \mathbb{N}$.)

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $M > 0$. Prove that the following two conditions are equivalent:
(a) $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$,
(b) f is absolutely continuous function which satisfies $|f'(x)| \le M$ for all $x \in \mathbb{R}$.

4. Let m be the Lebesgue measure on $\mathbb{R}$ and m^* the Lebesgue outer measure. For $A \subseteq \mathbb{R}$, define

$$m_*(A) = \sup\{m(K) : K \subseteq A, K \text{ compact}\}.$$

Prove that if $m^*(A) = m_*(A)$ and $m^*(A) < \infty$, then A is Lebesgue measurable.

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1. Let f_n be a sequence of measurable functions on (X, μ) and $f_n \geq 0$. Assume that $\int f_n \, d\mu = 1$ for all $n \in \mathbb{N}$. Prove that $\limsup_{n\to\infty} f_n(x)^{1/n} \leq 1$ for μ-a.e. x.

2. Let $x = 0.n_1 n_2 n_3 \ldots$ be the decimal expansion of $x \in [0,1]$, and define the function $f(x) = \min_i n_i$. Prove that f is measurable and constant a.e. (When a number has two different representations, we use the one with repeated zeros.)

3. Let f be a continous function on $[0,1]$. Let $F(x) = \sup_{0\leq y\leq x} f(y)$ for $x \in [0,1]$.
a) Is F a Borel function?
b) Prove that $S = \{x \in (0,1] : f(x) > f(y) \text{ for all } 0 \leq y < x\}$ is Borel.

4. Let $f(x) = \sum_{n=1}^{\infty} a_n e^{-nx}$ for $x \geq 0$, where $a_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n < \infty$. Prove that $\sum_{n=1}^{\infty} n a_n < \infty$ if and only if the right-hand derivative of f exists at 0.

REAL ANALYSIS GRADUATE EXAM

Spring 2021

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1. Suppose that f is a bounded non-negative function on a measure space (X, μ) with $\mu(X) = \infty$. Prove that f is integrable if and only if

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \mu(\{x \in X : f(x) > 2^{-n}\}) < \infty.$$

2. Let f be a real valued function on $[0, 1]$. Prove that the set of points where f is continuous is Lebesgue measurable.

3. Let f be a Lebesgue integrable function on $\mathbb{R}$ and let $\beta \in (0, 1)$. Prove that

$$\int_0^{\infty} \frac{|f(x)|}{|x-a|^{\beta}}\, dx < \infty.$$

for a.e. $a \in \mathbb{R}$.

4. Let f_n be a sequence of real-valued function on $[a, b]$ so that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in [a, b]$. Let $V_a^b(f)$ be the total variation of f on $[a, b]$. Show that

$$V_a^b(f) \leq \liminf_{n\to\infty} V_a^b(f_n).$$

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1. Let f be integrable on $\mathbb{R}^d$ with respect to the Lebesgue measure m. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that A Lebesgue measurable and $m(A) < \delta$ imply $\int_A |f(x)|dx < \epsilon$.

2. Assume that f is integrable on $\mathbb{R}$ with respect to the Lebesgue measure. Prove that

$$\lim_{h \to 0} \int |f(x+h) - f(x)|\, dx = 0.$$

3. Assume that f and $\{f_n\}_{n=1}^{\infty}$ are Lebesgue measurable on $\mathbb{R}$ and that we have

$$\int |f(x) - f_n(x)|\, dx \leq \frac{C}{n^2}, \qquad n \in \mathbb{N},$$

for some constant $C \geq 0$. Prove that $f_n \to f$ a.e. as $n \to \infty$.

4. Assume that f is integrable on $\mathbb{R}$ with respect to the Lebesgue measure m and that $f(x) > 0$ a.e. Let E_k be a sequence of Lebesgue measurable sets in $[0,1]$ such that

$$\lim_{k \to \infty} \int_{E_k} f(x)\, dx = 0.$$

Prove that $\lim_{k \to \infty} m(E_k) = 0$.

REAL ANALYSIS GRADUATE EXAM

Spring 2022

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Prove that for almost all $x \in [0,1]$, with respect to the Lebesgue measure, there are at most finitely many rational numbers in a reduced form $\frac{p}{q}$, where $q \geq 2$, so that

$$\left| x - \frac{p}{q} \right| < \frac{1}{(q \log q)^2}.$$

(Hint: Consider intervals of lengths $1/(q \log q)^2$ centered at rational points p/q.)

2. Let $S \subseteq \mathbb{R}$ be closed, and let $f \in L^1([0,1], m)$, where m denotes the Lebesgue measure on $[0,1]$. Assume that for all measurable $E \subseteq [0,1]$ with $m(E) > 0$ we have

$$\frac{1}{m(E)} \int_E f(x)\, dx \in S.$$

Prove that $f(x) \in S$ for a.e. $x \in [0,1]$.

3. Evaluate the limit

$$\lim_n \int_0^1 \frac{1+nx}{(1+x)^n}\, dx.$$

4. Assume that (X, μ) is a finite measure space and $\{f_n\}_{n=1}^{\infty}$ a sequence of nonnegative measurable functions on X. Prove that $f_n \to 0$ in measure if and only if

$$\lim_n \int_X \frac{f_n(x)(2 + f_n(x))}{(1 + f_n(x))^2}\, d\mu(x) = 0.$$

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Fall 2022

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1. Let f be Lebesgue measurable on $\mathbb{R}$ and $E \subset \mathbb{R}$ be measurable so that $0 < A = \int_E f(x)\,dx < \infty$. Show that for every $t \in (0,1)$ there exists a measurable set $E_t \subset E$ so that $\int_{E_t} f(x)\,dx = tA$.

2. Assume that $f \in \mathscr{L}^1(\mathbb{R})$ and let $F(t) = \int f(x)e^{itx}\,dx$. Prove that $F : \mathbb{R} \to \mathbb{R}$ is continuous and that $\lim_{t\to\infty} F(t) = \lim_{t\to-\infty} F(t) = 0$.

3. Compute the following limit and justify your calculations

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x^2}{n}\right)^{-(n+1)} dx.$$

4. For $f \in \mathscr{L}^1(\mathbb{R})$ consider the maximal function

$$\mathcal{M}f(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|\,dt.$$

Prove that there exists a constant $A > 0$ so that for any $\alpha > 0$

$$m(\mathcal{M}f > \alpha) \le \frac{A}{\alpha} |f|_{\mathscr{L}^1}$$

where m is the Lebesgue measure on $\mathbb{R}$.

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1. Let f be Lebesgue measurable in $[0,1]$ with $f(x) > 0$ for a.e. x. Suppose E_k is a sequence of measurable sets in $[0,1]$ with the property $\int_{E_k} f(x)\,dx \to 0$ as $k \to \infty$. Prove that $m(E_k) \to 0$ as $k \to \infty$.

2. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions on $[0,1]$ such that
(i) $|g_n(x)| \leq C$ a.e. $x \in [0,1]$ for some $C < \infty$, and
(ii) $\lim_{n\to\infty} \int_0^a g_n(x)\,dx = 0$ for all $a \in [0,1]$.
Prove that for all $f \in \mathcal{L}^1([0,1])$ one has

$$\lim_{n\to\infty} \int_0^1 f(x) g_n(x)\,dx = 0.$$

3. Suppose E is a measurable subset of $[0,1]$ with the Lebesgue measure $m(E) = \frac{99}{100}$. Show that there exists a number $x \in [0,1]$ such that for all $r \in (0,1)$ one has

$$m(E \cap (x-r, x+r)) \geq \frac{r}{4}.$$

(Hint: Use the Hardy-Littlewood inequality $m(\{x \in \mathbb{R} : Mf(x) \geq \alpha\}) \leq \frac{3}{\alpha}\|f\|_{\mathcal{L}^1}$, where Mf denotes the Hardy-Littlewood maximal function of f.)

4. Let $f\colon [0,1] \to [0,1]$ be Lebesgue measurable. Prove that for every $M > 0$, there exists $a \in [0,1]$ such that

$$\int_0^1 \frac{dx}{|f(x) - a|} \geq M.$$

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1. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a nonnegative function which is integrable with respect to a measure μ on $\mathbb{R}$. Prove that for every $\epsilon > 0$, there exists a μ-measurable set $A \subseteq \mathbb{R}$ such that $\mu(A) < \infty$ and $\int_A f\, d\mu \geq \int f\, d\mu - \epsilon$.

2. Does there exist the limit

$$\lim_{n\to\infty} \int_0^1 \frac{(3x-1)^{2n}\, dx}{1+(3x-1)^{2n}}?$$

Prove your assertion.

3. Assume that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is such that $\int |f|\, dx > 0$. Prove that the maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r(x)} |f(y)|\, dy$$

does not belong to $L^1(\mathbb{R}^n)$.

4. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable. Prove that there exists a Borel measurable function $g\colon \mathbb{R} \to \mathbb{R}$ such that $f = g$ Lebesgue-a.e.

REAL ANALYSIS GRADUATE EXAM

Spring 2024

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1. Consider a sequence $f_n = f_n(x), x \geq 0, n = 1, 2, \dots,$ of continuously differentiable functions with corresponding derivatives f_n'. Assume that $f_n(0) = 0$ and

$$\lim_{n\to\infty} \int_0^{+\infty} |f_n'(x)|^2 \, dx = 0.$$

Is is true of false that

$$\lim_{n\to\infty} \sup_{x\geq 0} |f_n(x)| = 0?$$

Explain your conclusion by proof or counterexample.

2. Let $(X, \mathcal{A}, \mu)$ be a complete measure space and let f be a non-negative integrable function on X. Put $b(t) = \mu(\{x \in X : f(x) \geq t\})$. Show that

$$\int_X f \, d\mu = \int_0^\infty b(t) \, dt.$$

3. Let X be a compact metric space, μ a finite non-negative Borel measure on X. Suppose $\mu(\{x\}) = 0$ for all $x \in X$. Prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mu(X) < \varepsilon$ whenever E is a Borel set in X of diameter $< \delta$.

4. (i) Assume that $f \in \mathscr{L}^1(0, \infty)$. Prove that

$$g(x) = \int_0^\infty \frac{f(y)}{x + y} \, dy$$

is differentiable at every $x > 0$.
(ii) In (i) find an example of $f \in \mathscr{L}^1(0, \infty)$ such that $g : [0, \infty) \to \mathbb{R}$ is not differentiable at $x = 0$.

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1. Evaluate the limit and justify the steps along the way:

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{x^2-5x+1}{1+x^{6n}}\,dx.$$

2. Let f be a non-decreasing function on $[0,1]$. Assume that f is differentiable a.e.. Prove that $\int_0^1 f'(x)\,dx \le f(1)-f(0)$.

3. Assume that $E\subset\mathbb{R}$ is of finite Lebesgue measure, and let $f\in\mathscr{L}^1(\mathbb{R})$. Prove that

$$\lim_{t\to\infty}\int_E f(x+t)\,dx = 0.$$

4. Let f, g be real valued integrable functions on a complete measure space $(X,\mathcal{B},\mu)$ and define

$$F_t=\{x\in X : f(x)>t\},\qquad G_t=\{x\in X : g(x)>t\}.$$

Prove that

$$\int_X |f-g|\,d\mu=\int_{-\infty}^{\infty}\mu((F_t\setminus G_t)\cup(G_t\setminus F_t))\,dt,$$

where $F_t\setminus G_t=\{x\in F_t : x\notin G_t\}$ and similarly $G_t\setminus F_t=\{x\in G_t : x\notin F_t\}$.