Geometry and Topology Graduate Exam
Fall 2023

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Show that for any $n \geq 1$, the subset $\mathrm{SL}_n(\mathbb{R}) \subset \mathbb{R}^{n\times n}$ consisting of matrices of determinant 1 is a smooth submanifold.

Problem 2. Let $\alpha \in \Omega^2(\mathbb{R}^3)$ be the 2-form defined by

$$\alpha = (x^3 + y + z)\,\mathrm{d}y \wedge \mathrm{d}z - (x + y^3 + z)\,\mathrm{d}x \wedge \mathrm{d}z + (x + y + z^3)\,\mathrm{d}x \wedge \mathrm{d}y.$$

Compute the integral

$$\int_{\mathrm{S}^2} \alpha$$

over the unit sphere $\mathrm{S}^2 \subset \mathbb{R}^3$ (endowed with a fixed orientation of your choosing).
Hint: Recall that, for the spherical coordinates (r, θ, φ) *with* $x = r\cos\theta\cos\varphi$, $y = r\sin\theta\cos\varphi$, *and* $z = r\sin\varphi$, *we have* $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = r^2\cos\varphi\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\varphi$.

Problem 3. Let X be a manifold with $\pi_2(X, x) = 0$ for all $x \in X$. Is it necessarily the case that $\mathrm{H}_2(X; \mathbb{Z}) = 0$ as well? Explain.

Problem 4. What are the integral homology groups of $\mathrm{S}^1 \vee \mathrm{S}^2 \vee \mathrm{S}^3 \vee \mathrm{S}^4$?

Problem 5. Let V be a smooth vector-field on a closed manifold M and let $\varphi : \mathbb{R} \times M \to M$ be the flow generated by V. Consider the quotient space X by the flow, i.e.

$$X = M/\!\sim \qquad \text{where} \qquad p \sim q \text{ if and only if } q = \varphi(t, p) \text{ for some time } t.$$

Prove or give a counter-example to the following statements.

(a) X is compact.

(b) X is a closed manifold.

Problem 6. Give an example of a covering space $X \to Y$ which is not a regular covering space.

Problem 7. Show that the Cantor set does not admit a CW complex structure.