REAL ANALYSIS GRADUATE EXAM

Fall 2023

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a nonnegative function which is integrable with respect to a measure μ on $\mathbb{R}$. Prove that for every $\epsilon > 0$, there exists a μ-measurable set $A \subseteq \mathbb{R}$ such that $\mu(A) < \infty$ and $\int_A f\,d\mu \geq \int f\,d\mu - \epsilon$.

2. Does there exist the limit

$$\lim_{n\to\infty} \int_0^1 \frac{(3x-1)^{2n}\,dx}{1+(3x-1)^{2n}}?$$

Prove your assertion.

3. Assume that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is such that $\int |f|\,dx > 0$. Prove that the maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r(x)} |f(y)|\,dy$$

does not belong to $L^1(\mathbb{R}^n)$.

4. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable. Prove that there exists a Borel measurable function $g\colon \mathbb{R} \to \mathbb{R}$ such that $f = g$ Lebesgue-a.e.