Algebra qualifying exam, August 2023
Justify all arguments completely. Reference specific results whenever possible.

1. Let $R = C_{\mathrm{an}}(\mathbb{C})$ denote the ring of complex analytic functions on $\mathbb{C}$. We note that R is a domain. Prove one of the following statements:

(a) The ring $R[t]$ of polynomials with coefficients in R *is* a PID.
(b) The ring $R[t]$ *is not* a PID.

2. Let A be an Artinian ring, and let M be an A-module which is annihilated by all nilpotent ideals in A. Prove that M is a semisimple A-module.

3. (i) Determine the radical of the ideal

$$I = \big(y^2 - 1,\ x^2 - (y+1)x + 1\big)$$

in $\mathbb{C}[x, y]$. You may write your answer in any form which allows you to easily see if a given polynomial $f(x, y)$ is in the radical or not.
(ii) Determine if the inclusion $I \subseteq \sqrt{I}$ is an equality.

4. Classify all groups of order 50.

5. Consider the polynomial $p(x) = x^{16} - \alpha x^{10} - \alpha x^6 + \alpha^2$, for α non-algebraic over $\mathbb{Q}$. Take $F = \mathbb{Q}(\alpha, \zeta)$, where $\zeta = e^{2\pi i/15}$.

(i) Is $p(x)$ is irreducible over F?
(ii) Determine the Galois group $\mathrm{Gal}(K/F)$ for the splitting field K of $p(x)$ over F.

6. For a given group G, let G' denote the subgroup generated by the commutators $[a, b] = a^{-1}b^{-1}ab$ in G. Explicitly,

$$G' = \langle a^{-1}b^{-1}ab :\ a, b \in G\rangle \ \subseteq\ G.$$

(i) Prove that G' is normal in G, and that G/G' is abelian.
(ii) Show that if H is normal in G and G/H is abelian, then H contains G'.
(iii) Prove that $S_5' = A_5$.