Numerical Analysis Preliminary Examination Fall 2023

August 9, 2023

Problem 1.

Let $T \in \mathbb{C}^{n\times n}$ be given and let $\rho(T)$ denote the spectral radius of T.

(a) Show that for any induced norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, $\|T\| \geq \rho(T)$.

(b) Given $\varepsilon > 0$, show there exists an induced norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$ for which $\|T\| \leq \rho(T)+\varepsilon$.

Hint:

- *Recall that if $\|\cdot\|_*$ is an induced norm on matrices and S is invertible then the matrix function defined by $\|A\| = \|SAS^{-1}\|_*$ is also an induced norm. (You do not need to show this.)*
- *First show this the result is true when T is a Jordan block*

$$T = J_n(\lambda) = \begin{bmatrix} \lambda & 1 & \dots & 0 & 0 \\ 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{bmatrix}$$

by considering the norm defined by $\|A\| = \|D(n,\varepsilon)AD(n,\varepsilon)^{-1}\|_1$ where

$$D(n,\varepsilon) = \begin{bmatrix} 1/\varepsilon & 0 & \dots & 0 \\ 0 & 1/\varepsilon^2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1/\varepsilon^n \end{bmatrix}.$$

- *Then show it is true for general $T \in \mathbb{C}^{n\times n}$.*

(c) Show that for any induced norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$,

$$\lim_{k\to\infty} \|T^k\|^{1/k} = \rho(T).$$

Problem 2.

Consider the following $Ax = b$ Least Squares Problem (LSP).

$$\begin{bmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 15 \\ 9 \end{bmatrix}$$

For each of the following questions, set up the equations needed to solve the problem, evaluate the matrices and vectors involved and work a few calculations to illustrate that you know the steps required to solve the problem. It is not necessary to perform all the calculations.

(a) Solve the LSP above using the Normal equations.

(b) Use the Gram-Schmidt method to find a full QR factorization of A and use this factorization to solve the LSP.

(c) Use Householder reflectors to find a full QR factorization of A. *Hint: a Householder reflector H is a matrix of the form $H = I - 2vv^T/v^Tv$ where $v \neq 0$.* Are the factorizations found in parts (b) and (c) necessarily equal? Explain.

Problem 3.

Let $A \in \mathbb{C}^{m\times n}$ and consider the following system of matrix equations in the unknown matrix $X \in \mathbb{C}^{n\times m}$:

$$\begin{aligned} AXA &= A \\ XAX &= X \\ (AX)^* &= AX \\ (XA)^* &= XA, \end{aligned} \tag{1}$$

where $*$ denotes conjugate transpose.

(a) Show that if the system (1) has a solution, $X = A^+ \in \mathbb{C}^{n\times m}$, then it is unique.

(b) Use the singular value decomposition of A to show that in fact there exists a (unique) solution $A^+ \in \mathbb{C}^{n\times m}$ to the system (1).

(c) Show that if $b \in \mathbb{C}^m$ and $\operatorname{rank}(A) = n$, then $x = A^+b$ is the least squares solution to the system $Ax = b$.

(d) Show that if the system $Ax = b$ has at least one solution, then $x = A^+b$ is the solution of minimum Euclidean norm.