Spring 2012 Complex Analysis Exam

Answer all four questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Suppose $a > 0$. Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+1)}\,dx,$$
being careful to justify your methods.

2. Let $f(z)$ be analytic for $0 < |z| < 1$. Assume there are $C > 0$ and $m \geq 1$ such that
$$|f^{(m)}(z)| \leq \frac{C}{|z|^m}, 0 < |z| < 1.$$
Show that f has a removable singularity at $z = 0$.

3. Let $D \subseteq \mathbf{C}$ be a connected open subset and let (u_n) be a sequence of harmonic functions $u_n : D \to (0, \infty)$. Show that if $u_n(z_0) \to 0$ for some $z_0 \in D$, then $u_n \to 0$ uniformly on compact subsets of D.

4. Let D be the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane, and define $\Omega = D \setminus [0, 1]$. Find a conformal mapping of Ω onto D. You may give your answer as the composition of several mappings, so long as each mapping is precisely described.

COMPLEX ANALYSIS GRADUATE EXAM

Fall 2012

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate the integral

$$\int_0^\infty \frac{dx}{1+x^n}, n \geq 2,$$

being careful to justify your methods.

2. Find the Laurent series expansion for

$$\frac{1}{z(z+1)}$$

valid in $\{1 < |z-1| < 2\}$.

3. Suppose that f is an entire function and that there is a bounded sequence of distinct real numbers $a_1, a_2, a_3, \ldots$ such that $f(a_k)$ is real for each k. Show that $f(x)$ is real for all real x.

4. Suppose

$$f_n(z) = \sum_{k=0}^{n} \frac{1}{k!z^k}, z \neq 0$$

and let $\varepsilon > 0$. Show that for large enough n, all the zeros of f_n are in the disk $D(0, \varepsilon)$ with center 0 and radius ε.

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2013

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate

$$\int_0^\infty \frac{x^{1/3}}{1+x^4}dx$$

being careful to justify your answer.

2. Assume that f is an entire function such that

$$|f(z)| \geq \frac{1}{1+|z|} \text{ for all } z \in \mathbb{C}.$$

Prove that f is a constant function.

3. Let $f_n, n \geq 1$, be a sequence of holomorphic functions on an open connected set D such that $|f_n(z)| \leq 1$ for all $z \in D, n \geq 1$. Let $A \subseteq D$ be the set of all $z \in D$ for which the limit $\lim_n f_n(z)$ exists.

Show that if A has an accumulation point in D, then there exists a holomorphic function f on D such that $f_n \to f$ uniformly on every compact subset of D as $n \to \infty$.

4. Let $f(z)$ be meromorphic on $\mathbb{C}$, holomorphic for $\operatorname{Re} z > 0$ and such that $f(z+1) = zf(z)$ in its domain with $f(1) = 1$.

Show that f has the first order poles at $0, -1, -2, \ldots$, and find the residues of f at these points.

COMPLEX ANALYSIS GRADUATE EXAM

Fall 2013

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Compute
$$\int_0^\infty \frac{\log^2 x}{1+x^2}\,dx.$$

2. Find the number of *distinct* zeros of $f(z) = z^6 + (10-i)z^4 + 1$ inside $(-1,1)\times(-1,1)$.

3. Suppose that f is holomorphic in a neighborhood U of $a \in \mathbb{C}$. Consider the following two statements:

(i) There exist two sequences $\{z_k\}_{k=1}^\infty$ and $\{w_k\}_{k=1}^\infty$ in $U\backslash\{a\}$ converging to a such that $z_k \neq w_k$ and $f(z_k) = f(w_k)$ for all $k \in \mathbb{N}$.

(ii) $f'(a) = 0$.

Determine whether either of the statements implies the other one. In each case justify your answer with a proof or a counterexample.

4. Let f be analytic in an open set $U \subseteq \mathbb{C}$, and let $K \subseteq U$ be compact. Show that there exists a constant C depending on U and K such that
$$|f(z)| \leq C\left(\int_U |f|^2\right)^{1/2}$$
for all $z \in K$.

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2014

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. For $a > 0$, evaluate the integral

$$\int_0^\infty \frac{\log x}{(a+x)^3}\, dx,$$

being careful to justify your methods.

2. Find a conformal mapping of the region $\{z : |z| > 1\} \setminus (1, \infty)$ onto the open unit disc $\{z : |z| < 1\}$. You may give your answer as the composition of several mappings, so long as each mapping is precisely described.

3. Suppose that f_n are analytic functions on a connected open set $U \subset \mathbb{C}$ and that $f_n \to f$ uniformly on compact subsets of U. In each case indicate the main steps in the proofs of the following standard results.

(i) f is analytic in U;

(ii) $f_n' \to f'$ uniformly on compact subsets of U;

(iii) if $f_n(z) \neq 0$ for all n and all $z \in U$, then either $f(z) \neq 0$ for all $z \in U$ or else $f \equiv 0$.

4. (a) Suppose that f is analytic on the open unit disc $\{z : |z| < 1\}$ and that there exists a constant M such that $|f^k(0)| \leq k^4 M^k$ for all $k \geq 0$. Show that f can be extended to be analytic on $\mathbb{C}$.

(b) Suppose that f is analytic on the open unit disc $\{z : |z| < 1\}$ and that there exists a constant $M > 1$ such that $|f(1/k)| \leq M^{-k}$ for all $k \geq 1$. Show that f is identically zero.

COMPLEX ANALYSIS GRADUATE EXAM

Fall 2014

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $a > 1$. Compute

$$\int_0^\pi \frac{d\theta}{a + \cos\theta}$$

being careful to justify your methods.

2. Find the number of zeros, counting multiplicity, of $z^8 - z^3 + 10$ inside the first quadrant $\{z \in \mathbb{C} : \mathbb{R}\mathrm{e}\, z > 0, \mathbb{I}\mathrm{m}\, z > 0\}$.

3. Assume that $f(z)$ and $g(z)$ are holomorphic in a punctured neighborhood of $z_0 \in \mathbb{C}$. Prove that if f has an essential singularity at z_0 and g has a pole at z_0, then $f(z)g(z)$ has an essential singularity at z_0.

4. (i) Suppose that f is holomorphic on $\mathbb{C}$ and assume that the imaginary part of f is bounded. Prove that f is constant.
(ii) Suppose that f and g are holomorphic on $\mathbb{C}$ and that $|f(z)| \le |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists $\lambda \in \mathbb{C}$ such that $f = \lambda g$.

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2015

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate the integral

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx,$$

being careful to justify your answer.

2. Determine the number of roots of $f(z) = z^9 + z^6 + z^5 + 8z^3 + 1$ inside the annulus $1 < |z| < 2$.

3. Suppose that f is holomorphic on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and suppose that for $z \in \mathbb{D}$ one has $\Re(f(z)) > 0$ and $f(0) = 1$. Prove that $|f(z)| \leq \frac{1+|z|}{1-|z|}$ for all $z \in \mathbb{D}$.

4. For $a_n = 1 - \frac{1}{n^2}$, let

$$f(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - a_n z}.$$

(1) Show that f defines a holomorphic function on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
(2) Prove that f does not have an analytic continuation to any larger disk $\{z \in \mathbb{C} : |z| < r\}$ for some $r > 1$.

COMPLEX ANALYSIS GRADUATE EXAM

Fall 2016

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $A = \{z \in \mathbb{C} : r < |z| < R\}$ for some $0 < r < R < \infty$. Prove that $f(z) = 1/z$ cannot be uniformly approximated in A by complex polynomials.

2. Let $D = \mathbb{C} \backslash [-1, 1]$. Prove that $f(z) = z^2 - 1$, for $z \in D$, has an analytic square root but does not have an analytic logarithm.

3. Evaluate

$$\int_0^\infty \frac{\log x}{1 + x^2}\, dx.$$

4. Show that the range of a nonconstant entire function is dense in $\mathbb{C}$.

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2015

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate the integral

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx,$$

being careful to justify your answer.

2. Let

$$f(z) = \sum_{n=0}^{\infty} z^{n!}.$$

(i) Show that f is analytic in the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.
(ii) Show that f can not be analytically continued to any open set properly containing D. (Hint: First consider $z = r^{2\pi i p/q}$ where p and q are integers.)

3. Let A be an open subset of $\mathbb{C}$, and suppose $u(x,y)$ is a twice continuously differentiable harmonic function on A.
(i) Show that if A is simply connected, then there exists an analytic function f on A such that $u = \mathbb{R}\mathrm{e}\, f$. (Hint: First find g so that $\partial u/\partial x = \mathbb{R}\mathrm{e}\, g$.)
(ii) Find f explicitly when $A = \mathbb{C}$ and $u(x,y) = e^x \cos y + xy$.
(iii) Give an example in which A is not simply connected and f as in (i) does not exist.

4. Determine whether it is possible for a function f to be analytic in a neighborhood of 0 and take the values $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \frac{1}{8}, \frac{1}{8}, \ldots$ at the points $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8} \ldots$.

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2016

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $a \in \mathbb{C}$ be such that $0 < |a| < 1$, and set

$$f(z) = \frac{1 - z^2}{z^2 - (a + \frac{1}{a})z + 1}.$$

Find the Laurent expansion of f in a neighborhood of the unit circle $|z| = 1$.

2. Let $a \in \mathbb{C}$ be such that $0 < |a| < 1$ and let $n \in \mathbb{N}$. Show that $e^z(z-1)^n = a$ has exactly n simple roots in the half-plane $\{z \in \mathbb{C} : \mathbb{R}\mathrm{e}\, z > 0\}$.

3. Evaluate

$$\int_0^\infty \frac{\log^2 x}{1 + x^2}\, dx.$$

4. Denote $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Assume that $f\colon \mathbb{D} \to \mathbb{D}$ is analytic. Show that if $z_1 \neq z_2$ are fixed points of f in $\mathbb{D}$, then f is the identity map.

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2017

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let γ be a circle with radius 1 and center 0 with positive direction of integration. Compute

$$\int_\gamma e^{1/z}\, dz.$$

2. Assume that f is a holomorphic function on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ satisfying

$$f(z)^3 = \overline{f(z)}, \qquad \forall z \in D.$$

Prove that f is a constant.

3. Let $f(z) = \sum_{n=1}^{\infty} z^{n!}$.
(i) Show that $f(z)$ is holomorphic in the unit disk $D = \{z \in \mathbb{C} : |z < 1\}$.
(ii) Show that f does not have any holomorphic extension, that is, there exists no g holomorphic on some open set $U \supseteq D$ such that $U \neq D$ and $f = g|D$. (Hint: Consider $e^{i\theta}$ where θ is rational.)

4. Let f be an entire function such that

$$f(z + m + ni) = f(z), \qquad \forall z \in \mathbb{C}, \qquad \forall m, n \in \mathbb{Z}.$$

Prove that f is a constant function.

COMPLEX ANALYSIS GRADUATE EXAM

Fall 2017

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $f(z) = u(z) + iv(z)$ be an entire function and assume that $|u(z)| \geq |v(z)|$ $\forall z \in \mathbb{C}$. Show that f is a constant.

2. Let $\alpha \in (0, 1)$ and $n \in \mathbb{N}$. Prove that the equation $e^z(z-1)^n = \alpha$ has exactly n *simple* roots in the right half plane $\{z : \Re(z) > 0\}$.

3. Evaluate the integral

$$\int_0^{2\pi} \frac{dt}{\cos t - 2}.$$

4. Write an entire function f which has the simple zeroes $1, 4, 9, 16, 25, \ldots$ and has no other zeroes.

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2018

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Show that there is no holomorphic function f in $\mathbb{D} = \{|z| < 1\}$ so that $|f(z_n)| \to \infty$ whenever $|z_n| \to 1$ ($z_n \in \mathbb{D}$).

2. Assume that f is analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Prove that f is odd if and only if all the terms in the Taylor series for f at $z_0 = 0$ have only odd powers.

3. Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2 + 1}\, dx$$

4. Let $\mathbb{D} = \{|z| < 1\}$ be the open unit disk and $\bar{\mathbb{D}}$ its closure. Let $f : \mathbb{D} \to \mathbb{C}$ be analytic on $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$. Assume that f takes only real values on $\partial\bar{\mathbb{D}} = \{|z| = 1\}$. Prove that f is constant.

COMPLEX ANALYSIS

Fall 2018

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $a > 0$. Compute

$$\int_0^\pi \frac{d\theta}{a^2 + \sin^2\theta}.$$

2. Find the number of solutions of the equation $z - 2 - e^{-z} = 0$ in $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

3. Let $\Omega \neq \mathbb{C}$ be simply connected and let for any $c \in \Omega$, the mapping $\phi_c : \Omega \to \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be conformal so that $\phi_c(c) = 0$. Let $g_c(z) = \log|\phi_c(z)|$, $z \in \Omega \setminus \{c\}$.

Show that $g_a(b) = g_b(a)$ for any distinct $a, b \in \Omega$.

4. Let $a \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and

$$f_a(z) = \frac{a - z}{1 - \bar{a}z}, z \in \bar{\mathbb{D}}.$$

Show that f_a is a holomorphic bijective mapping of $\mathbb{D}$ onto $\mathbb{D}$ which is its own inverse.

COMPLEX ANALYSIS

Spring 2019

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Find
$$\int_0^\infty \frac{\ln x}{1+x^4} dx.$$

2. Find the number of zeros of the polynomial $z^4 + 4z^2 + z + 1$ in the unit disk.

3. Describe all functions holomorphic in $\mathbb{D} = \{z : |z| < 1\}$ for which $f\left(\frac{1}{n}\right) = -\frac{1}{n^2}, n \in \mathbf{N}$.

4. Let f, g be entire functions such that $|f| \leq |g|$. Prove that $f = cg$ for some $c \in \mathbf{C}$.

COMPLEX ANALYSIS GRADUATE EXAM

Fall 2019

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Compute

$$\int_0^{2\pi} e^{-i\theta} \exp(e^{i\theta})\, d\theta.$$

2. Let $A = \{z : r < |z| < R\}$ for some $0 < r < R < \infty$. Prove that $f(z) = 1/z$ cannot be uniformly approximated in A by complex polynomials.

3. Assume that f is an entire function, which is not identically zero, and satisfies $\operatorname{Im} f(z) \cdot \operatorname{Im} z \geq 0$ for all $z \in \mathbb{C}$. Prove that $f'(z) \neq 0$ for all $z \in \mathbb{R}$.

4. Let f be an entire function which is non-constant. Show that $F(z) = e^{f(z)}$ has an essential singularity at ∞.

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2020

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Compute

$$\int_0^\infty \frac{\cos x \, dx}{(1+x^2)^2}.$$

2. Let $\Omega \subseteq \mathbb{C}$ be an open set such that $\{|z| \leq 1\} \subseteq \Omega$. Let $f_n \colon \Omega \to \mathbb{C}$ be a sequence of analytic functions which converge uniformly on compact subsets of Ω to $f \colon \Omega \to \mathbb{C}$. Assume $f(z) \neq 0$ for all $|z| = 1$. Prove that there exists $N \in \mathbb{N}$ so that f_n and f have the same number of zeroes in the unit disk $\{|z| < 1\}$ for all $n \geq N$.

3. Denote $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Let $f \colon \mathbb{D} \to \mathbb{D}$ be analytic and not identically z. Prove that f has at most one fixed point in $\mathbb{D}$. Is the same true if $\mathbb{D}$ is replaced by a simply connected bounded subset of $\mathbb{C}$? Prove or provide a counterexample.

4. Let f be an non-constant entire function. Prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

COMPLEX ANALYSIS GRADUATE EXAM

Fall 2020

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Compute

$$\int_{-\infty}^{\infty} \frac{e^{-itx}}{a^2 + x^2}\, dx,$$

where $a > 0$ and $t \in \mathbb{R}$, justifying all the steps.

2. Let $f(z)$ be a holomorphic function on the unit disk $D = \{z : |z| < 1\}$, and $f(0) = 0$. Show that $F(z) = \sum_{n=1}^{\infty} f(z^n)$, for $z \in D$, is holomorphic on D.

3. Is there a holomorphic function f on the unit disc so that $m_n \to \infty$ as $n \to \infty$, where

$$m_n = \inf\left\{ |f(z)| : 1 - \frac{1}{n} < |z| < 1 \right\}?$$

4. Denote by $D(0, r)$ the open disc around 0 with radius r. Let f, g be holomorphic functions from $D(0, 1)$ into an open set Ω. Assume that f is one-to-one, $f(D(0, 1)) = \Omega$ (thus f is bijective from $D(0, 1)$ to Ω), and $f(0) = g(0)$. Prove that

$$g(D(0, r)) \subseteq f(D(0, r)),$$

for all $r \in (0, 1)$.

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2021

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate

$$\int_{|z|=2} \frac{4z^7 - 1}{z^8 - 2z + 1} dz,$$

carefully justifying all the steps.

2. Let $\Omega \subseteq \mathbb{C}$ be an open set and $z_0 \in \Omega$. Suppose that f is holomorphic in $\Omega \backslash \{z_0\}$ and that z_0 is either a pole of order $m \in \mathbb{N}$ or a removable singularity for f whose removal results in z_0 being a zero of order m. Prove that z_0 is a first order pole of f'/f having residue either $-m$ or m.

3. Find a bijective analytic function which maps

$$\Omega = \left\{ z = x + iy \in \mathbb{C} : |z| < 1, \;\; y > -\frac{1}{\sqrt{2}} \right\}$$

to the unit disk $\mathbb{D} = \{z = x + iy \in \mathbb{C} : |z| < 1\}$.

4. Denote $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $f \colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function with two unequal fixed points (i.e., $f(a) = a$ and $f(b) = b$ with $a \neq b$). Prove that $f(z) = z$ for $z \in \mathbb{D}$.

COMPLEX ANALYSIS GRADUATE EXAM

Fall 2021

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate

$$\int_0^\infty \frac{dx}{x^{1/3}(1+x)},$$

carefully justifying all the steps.

2. Suppose that $\mathcal{F}$ is a normal family of analytic function on the open unit disk $\mathbb{D}$. Prove that $\mathcal{G} = \{f' : f \in \mathcal{F}\}$ is a normal family on $\mathbb{D}$.

3. Let $\Omega \subseteq \mathbb{C}$ be an open bounded connected set. Suppose that f is continuous on $\overline{\Omega}$, holomorphic in Ω, and that it satisfies $f(z) \neq 0$ for $z \in \overline{\Omega}$ and $|f| = C$ on $\partial\Omega$, where C is a constant.
(a) Prove that f is a constant function.
(b) Is the boundedness condition imposed on Ω essential?

4. Find all entire functions f such that $f((1+i)z) = f(z)$ for all $z \in \mathbb{C}$.

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2022

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Use residues to compute

$$\int_0^\infty \frac{dx}{1+x^{2022}}.$$

2. Map $\Omega = \{z \in \mathbb{C} : \mathbb{R}\mathrm{e}\, z > 0,\ \mathbb{I}\mathrm{m}\, z < 1\}$ conformally to $\mathbb{D}\backslash[0,1]$ (or the reverse), where $\mathbb{D}$ denotes the unit disk centered at 0.

3. Let $n \in \mathbb{N}$ and

$$f(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + 15z^8, \qquad z \in \mathbb{D},$$

where $\mathbb{D}$ is the unit disk centered at 0. Determine the number of zeros, with counted multiplicities, of f in $\mathbb{D}$.

4. Show that if f is entire and nowhere zero, then there exists an entire function g such that $f(z) = g(z)^2$ for all $z \in \mathbb{C}$.

COMPLEX ANALYSIS GRADUATE EXAM

Fall 2022

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate $\displaystyle\int_0^\infty \frac{\cos 8x}{x^2+1}\,dx.$

2. Does there exist a function $f(z)$ holomorphic in the unit disk $|z| < 1$ so that $\lim_{|z|\to 1} |f(z)| = \infty$?

3. Let $f(z)$ be holomorphic in the disk $|z| < 2$. Show that

$$\max_{|z|=1} \left| f(z) - \frac{1}{z} \right| \geq 1.$$

4. Map the region $\Omega = \{z \in \mathbb{C} : \Im(z) > 0\} \setminus \{1 + it : 0 < t \leq 1\}$ conformally to the upper half plane.

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2023

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $u(z)$ be a bounded harmonic function in $\mathbb{D} = \{|z| < 1\}$ such that we have the limits

$$\lim_{r \to 1^-} u(re^{i\varphi}) = \begin{cases} 1 & \text{if } 0 < \varphi < \pi \\ 0 & \text{if } \pi < \varphi < 2\pi. \end{cases}$$

Find $u(\frac{1}{2})$.

2. Let Γ be a closed curve in the right half plane that has index N with respect to the point 1. Find

$$\int_\Gamma e^{\frac{1}{z^2-1}} \sin \pi z \, dz.$$

3. Let $\mathcal{F}$ be that family of power series $\sum_{n=1}^{\infty} a_n z^n$ for which $|a_n| \le n$ for all $n \in \mathbb{N}$. Is $\mathcal{F}$ a normal family in the open unit disk $\mathbb{D} = \{|z| < 1\}$?

4. Let $r \in (0, 1)$. Prove that $p_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ has no zeros in $\{z \in \mathbb{C} : |z| < r\}$ provided n is sufficiently large.

COMPLEX ANALYSIS GRADUATE EXAM

Fall 2023

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate

$$\int_0^\infty \frac{dx}{x^4+1}.$$

2. Assume that $a \geq 1$ and $b > 2$. Prove that the equation

$$az^3 - z + b = e^{-z}(z+2)$$

has two solutions in $\{z : \mathbb{R}\mathrm{e}\, z > 0\}$.

3. Prove that the regions $\{z \in \mathbb{C} : 0 < |z| < 1\}$ and $\{z \in \mathbb{C} : 1 < |z| < 2\}$ are not conformally equivalent.

4. Assume that an entire function f maps a circle in a complex plane to $\mathbb{R}$. Prove that f is a constant.

COMPLEX ANALYSIS GRADUATE EXAM

Spring 2024

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $n \in \mathbb{N}$ and $\vartheta \in (0, \pi)$. Prove that

$$\frac{1}{2\pi i} \int_{|z|=2} \frac{z^n}{1 - 2z\cos\vartheta + z^2}\, dz = \frac{\sin(n\vartheta)}{\sin\vartheta},$$

where the integration contour is positively (counter clockwise) oriented.

2. Describe all the analytic functions f in $\mathbb{C} \setminus \{0\}$ with the property

$$|f(z)| \leq |z|^2 + \frac{1}{\sqrt{|z|}}$$

for all $0 < |z| < \infty$.

3. Assume $f_n, n = 1, 2, \ldots$, is analytic in $\mathbb{D} = \{|z| < 1\}$ and satisfies $|f_n| \leq 10$. Assume also that the limits $\lim_{n\to\infty} f_n(2^{-j})$ exist for every $j = 1, 2, \ldots$. Prove that the limits $\lim_{n\to\infty} f_n(z)$ exist for all $z \in \mathbb{D}$.

4. Consider the function

$$f(z) = \sum_{k=0}^{\infty} z^{2^k} = z + z^2 + z^4 + z^8 + z^{16} + \cdots .$$

Explain why f is analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and has the unit circle $\{z : |z| = 1\}$ as natural boundary, that is, cannot be continued analytically outside of $\mathbb{D}$.